ELEVENTH EDITION

Calculus with Applications

BRIEF VERSION

Lial • Greenwell • Ritchey

Calculus with Applications
Brief Version

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Calculus with Applications Brief Version

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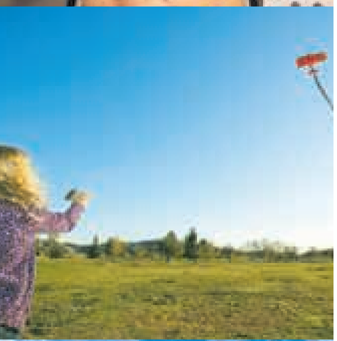

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Preface

Calculus with Applications, Brief Version is a thorough, applications-oriented text for students majoring in business, management, economics, or the life or social sciences. In addition to its clear exposition, this text consistently connects the mathematics to career and everyday-life situations. A prerequisite of two years of high school algebra is assumed. A greatly enhanced MyMathLab course, new applications and exercises, and other new learning tools make this 11th edition a rich learning resource for students.

Our Approach

Our main goal is to present applied calculus in a concise and meaningful way so that students can understand the full picture of the concepts they are learning and apply them to real-life situations. This is done through a variety of means.

Focus on Applications Making this course meaningful to students is critical to their success. Applications of the mathematics are integrated throughout the text in the exposition, the examples, the exercise sets, and the supplementary resources. We are constantly on the lookout for novel applications, and the text reflects our efforts to infuse it with relevance. Our research is showcased in the Index of Applications at the back of the book and the extended list of sources of real-world data on www.pearsonhighered.com/mathstatsresources. *Calculus with Applications, Brief Version* presents students with myriad opportunities to relate what they're learning to career situations through the *Apply It* question at the beginning of sections, the applied examples and exercises, and the *Extended Application* at the end of each chapter.

Pedagogy to Support Students Students need careful explanations of the mathematics along with examples presented in a clear and consistent manner. Additionally, students and instructors should have a means to assess the basic prerequisite skills needed for the course content. This can be done with the *Prerequisite Skills Diagnostic Test,* located just prior to Chapter R. If the diagnostic test reveals gaps in basic skills, students can find help right within the text. Further, *Warm-Up Exercises* are now included at the beginning of many exercise sets. Within MyMathLab are additional diagnostic tests (one per chapter), and remediation is automatically personalized to meet student needs. Students will appreciate the many annotated examples within the text, the *Your Turn* exercises that follow examples, the *For Review* references, and the wealth of learning resources within MyMathLab.

Beyond the Textbook Students want resources at their fingertips and, for them, that means digital access. So Pearson has developed a robust MyMathLab course for *Calculus with Applications, Brief Version*. MyMathLab has a well-established and well-documented track record of helping students succeed in mathematics. The MyMathLab online course for this text contains over 1800 exercises to challenge students and provides help when they need it. Students who learn best through video can view (and review) section- and example-level videos within MyMathLab. These and other resources are available to students as a unified and reliable tool for their success.

New to the Eleventh Edition

Based on our experience in the classroom along with feedback from many instructors across the country, the focus of this revision is to improve the clarity of the presentation and provide students with more opportunities to learn, practice, and apply what they've learned on their own. We do this both in the presentation of the content and in the new features added to the text.

New Features

- *Warm-Up Exercises* were added to many exercise sets to provide an opportunity for students to refresh key prerequisite skills at "point of use."
- Graphing calculator screens have been updated to reflect the TI-84 Plus C, which features color and a higher screen resolution. Additionally, the graphing calculator notes have been updated throughout.
- We added more "help text" annotations to examples. These notes, set in small blue type, appear next to the steps within worked-out examples and provide an additional aid for students with weaker algebra skills.
- For many years this text has featured enormous amounts of real data used in examples and exercises. The 11th edition will not disappoint in this area. We have added or updated 137 (18.3%) of the application exercises throughout the text.
- We updated exercises and examples based on user feedback and other factors. Of the 2720 exercises within the sections, 304 (11.2%) are new or updated. Of the 320 examples in the text, 48 (15%) are new or updated.
- MyMathLab contains a wealth of new resources to help students learn and to help you as you teach. Some resources were added or revised based on student usage of the *previous* edition of the MyMathLab course. For example, more exercises were added to those chapters and sections that are more widely assigned.
    - Hundreds of new exercises were added to the course to provide you with more options for assignments, including:
        - More application exercises throughout the text
        - *Setup & Solve* exercises that require students to specify how to set up a problem as well as solve it
        - Exercises that take advantage of the enhanced graphing tool
    - An Integrated Review version of the course contains preassigned diagnostic and remediation resources for key prerequisite skills. Skills Check Quizzes help diagnose gaps in skills prior to each chapter. MyMathLab then provides personalized help on only those skills that a student has not mastered.
    - The videos for the course have increased in number, type, and quality:
        - New videos feature more applications and more challenging examples.
        - In addition to full-length lecture videos, MyMathLab now includes assignable, shorter video clips that focus on a specific concept or example.
        - MathTalk Videos help motivate students by pointing out relevant connections to their majors—especially business. The videos feature Andrea Young from Ripon College (WI), a dynamic math professor (and actor!). The videos can be used as lecture starters or as part of homework assignments (in regular or flipped classes). Assignable exercises that accompany the videos help make these videos a part of homework assignments.
        - A Guide to Video-Based Instruction shows which exercises correspond to each video, making it easy to assess students after they watch an instructional video. This is perfect for flipped-classroom situations.
    - Learning Catalytics is a "bring your own device" student engagement, assessment, and classroom intelligence system. Students can use any web-enabled device—laptop, smartphone, or tablet—that they already have. Those with access to MyMathLab have instant access to Learning Catalytics and can log in using their MyMathLab username and password. With Learning Catalytics, you assess students in real time, using open-ended tasks to probe student understanding. It allows you to engage students by creating open-ended questions that ask for numerical, algebraic, textual, or graphical responses—or just simple multiple-choice. Learning Catalytics contains Pearson-created content for calculus so you can take advantage of this exciting technology immediately.

New and Revised Content

The chapters and sections in the text are in the same order as the previous edition, making it easy for users to transition to the new edition. In addition to revising exercises and examples throughout, updating and adding real-world data, we made the following changes:

Chapter R

- Added new *Your Turn* exercises to ensure that there is a student assessment for each major concept.
- Added more detail to R.2 on factoring perfect squares.

Chapter 1

- Rewrote the part of 1.1 involving graphing lines, emphasizing different methods for graphing.
- Rewrote 1.2 on supply, demand, break-even analysis, and equilibrium; giving formal definitions that match what students would see in business and economics courses. All of the business applications were revised, according to recommendations from reviewers, to be more in line with business texts. Also added a new Example 6 on finding a cost function.
- Added color for pedagogical reasons to make content easier to follow.

Chapter 2

- Updated the introduction to 2.1, rewriting it as an example to make it easier for students to reference the necessary skills to identify nonlinear functions, determine the domain and range, and estimate values from a graph.
- In 2.2, added another approach to graphing parabolas by splitting former Example 4 into two separate examples. The new Example 5 illustrates how to graph a parabola by first finding its characteristics (including orientation, intercepts, vertex, and axis of symmetry). The characteristics are highlighted in a box for easy reference.
- Added quadratic regression to 2.2. Example 9 includes a by-hand method and a method using technology.
- Rewrote Example 10 in 2.2, which illustrates translations and reflections of a graph, by breaking it into three parts. The first part is a basic transformation, and the ensuing parts build in complexity.
- Added the definition of a real root to 2.3 and added a Technology Note to illustrate how to use a graphing calculator to approximate the roots of higher degree polynomials.
- Added cubic regression to 2.3 (Example 5).

Chapter 3

- Added Caution note to 3.1 and added a new solution method to Example 9.
- Added new Example 2 to 3.3, using recent data.
- Updated Example 4 in 3.3 to use clearer wording.

Chapter 4

- Clarified the rules for differentiation in 4.1, 4.2, and 4.3 and added a new Example 8.
- Expanded Example 9 in 4.1 to include a new graph.
- Updated Example 10 in 4.1 and Example 4 in 4.5.

Chapter 5

- Added new examples to 5.2 (Example 3(c)) and 5.3 (Example 6(b)).
- Expanded Example 6(a) in 5.4 to show the inflection point.

Chapter 6

- Updated Example 3 in 6.1 to show an application of the concept.
- Modified examples in 6.2 (Example 3), 6.4 (Example 2), and 6.6 (Example 1).

Chapter 7

- Added annotations and comments to Example 10 in 7.1.
- Simplified Examples 1, 2, 3, and 6 in 7.2 and added annotations and comments.
- Added a "For Review" box to 7.3.
- Enlarged all small integral signs throughout the chapter for clarity.
- Updated Example 7 in 7.4 and Example 5 in 7.5.
- Added more explanation of the consumer surplus to 7.5.

Chapter 8

- Added annotations to several examples in 8.1 to denote steps in integration by parts.
- Revised the solutions to Examples 4 and 5 in 8.3, giving more detail and adding annotation to denote the steps in determining the accumulated amount of money flow.

Chapter 9

- Rewrote and expanded Exercise 8 in 9.1, on the Cobb-Douglas Production Function, emphasizing the interpretation of the solutions.
- Added three new exercises to 9.1 on exponential and logarithmic functions of several variables.
- Revised the solution to Example 4 in 9.3, giving more detail.
- Rewrote the solution to Example 3 in 9.4, illustrating how to find the extrema of a constrained function of one or more variables using a spreadsheet.

Features of *Calculus with Applications*

Chapter Opener

Each chapter opens with a quick introduction that relates to an application presented in the chapter.

Apply It

An Apply It question, typically at the start of a section, motivates the math content of the section by posing a real-world question that is then answered within the examples or exercises.

Teaching Tips

Teaching Tips are provided in the margins of the Annotated Instructor's Edition for those who are new to teaching this course.

For Review

For Review boxes are provided in the margin as appropriate, giving students just-in-time help with skills they should already know but may have forgotten. For Review comments sometimes include an explanation, while others refer students back to earlier parts of the book for a more thorough review.

> **FOR REVIEW**
> Recall that $e^x > 0$ for all x, so there can never be a solution to $e^{g(x)} = 0$ for any function $g(x)$.

Caution

Caution notes provide students with a quick "heads up" to common difficulties and errors.

CAUTION Notice from Example 5(c) that $g(x + h)$ is *not* the same as $g(x) + h$, which equals $-x^2 + 4x - 5 + h$. There is a significant difference between applying a function to the quantity $x + h$ and applying a function to x and adding h afterward.

Your Turn Exercises

These exercises follow selected examples and provide students with an easy way to quickly stop and check their understanding. Answers are provided at the end of the section's exercises.

Technology Notes

Material on graphing calculators or Microsoft Excel is clearly labeled to make it easier for instructors to use this material (or not).

- **New** The figures depicting calculator screens now reflect the TI-84 Plus C, which features color and higher pixel counts.

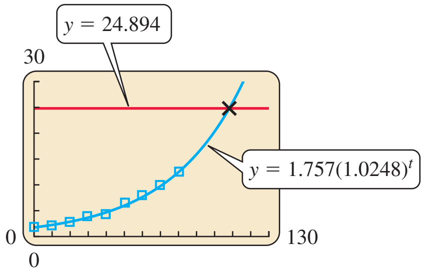

FIGURE 56

Exercise Sets

Basic exercises are followed by an Applications section, which is grouped by subheads such as "Business and Economics." Other types of exercises include the following:

- **New** **Warm-Up** exercises at the beginning of most sections provide a chance for students to refresh the key prerequisite skills needed for the section's exercises.
- **Connections** exercises integrate topics presented in different sections or chapters and are indicated with .
- **Writing** exercises, labeled with , provide students with an opportunity to explain important mathematical ideas.
- **Technology** exercises are labeled for graphing calculator and for spreadsheets.
- Exercises that are particularly challenging are denoted with a + in the Annotated Instructor's Edition only.
- The Annotated Instructor's Edition contains most answers right on the page. Overflow answers are at the back of the book.

Chapter Summary and Review

- The end-of-chapter **Summary** provides students with a quick summary of the key ideas of the chapter followed by a list of key definitions, terms, and examples.
- Chapter **Review Exercises** include Concept Check exercises and an ample set of Practice and Exploration exercises. This arrangement provides students with a comprehensive set of exercises to prepare for chapter exams.

Extended Applications

- Extended Applications are provided at the end of every chapter as in-depth applied exercises to help stimulate student interest. These activities can be completed individually or as a group project. Additional Extended Applications for the text can be found online at www.pearsonhighered.com/mathstatsresources.

Supplements

FOR STUDENTS

Student's Solutions Manual (in print and electronically within MyMathLab)

- Provides detailed solutions to all odd-numbered text exercises and sample chapter tests with answers.
- ISBN 0133864537 / 9780133864533

Graphing Calculator Manual for Applied Mathematics (downloadable)

- Contains detailed instruction for using the TI-83/ TI-83+/ TI-84+C
- Instructions are organized by topic.
- Downloadable from within MyMathLab

Excel Spreadsheet Manual for Applied Mathematics (downloadable)

- Contains detailed instruction for using Excel 2013
- Instructions are organized by topic.
- Downloadable from within MyMathLab

FOR INSTRUCTORS

Annotated Instructor's Edition

- Includes all the answers, usually on the same page as the exercises, for quick reference
- Challenge problems labeled with a + (plus sign)
- Numerous teaching tips
- ISBN 0321998774 / 9780321998774

Instructor's Solutions and Resource Manual (Download Only)

- Provides complete solutions to all exercises, two versions of a pre-test and final exam, and teaching tips.
- Available to qualified instructors within MyMathLab or through the Pearson Instructor Resource Center (www.pearsonhighered.com/irc).
- ISBN 0133864650 / 9780133864656

PowerPoint Presentations (Download Only)

- Includes lecture content and key graphics from the book.
- Available to qualified instructors within MyMathLab or through the Pearson Instructor Resource Center (www.pearsonhighered.com/irc).
- ISBN 0133983862 / 9780133983869

TestGen Computerized Test Bank

- TestGen® (www.pearsoned.com/testgen) enables instructors to build, edit, print, and administer tests using a computerized bank of questions developed to cover all the objectives of the text.
- TestGen is algorithmically based, allowing instructors to create multiple but equivalent versions of the same question or test with the click of a button. Instructors can also modify test bank questions or add new questions.
- The software and testbank are available to qualified instructors within MyMathLab or through the Pearson Instructor Resource Center (www.pearsonhighered.com/irc).
- ISBN 0133864626 / 9780133864625

MyMathLab® Online Course (access code required)

MyMathLab delivers **proven results** in helping individual students succeed.

- MyMathLab has a consistently positive impact on the quality of learning in higher education math instruction. MyMathLab can be successfully implemented in any environment—lab-based, hybrid, fully online, traditional—and demonstrates the quantifiable difference that integrated usage has on student retention, subsequent success, and overall achievement.
- MyMathLab's comprehensive online gradebook automatically tracks your students' results on tests, quizzes, homework, and in the study plan. You can use the gradebook to quickly

intervene if your students have trouble, or to provide positive feedback on a job well done. The data within MyMathLab's gradebook are easily exported to a variety of spreadsheet programs.

MyMathLab provides **engaging experiences** that personalize, stimulate, and measure learning for each student.

- **Personalized Learning:** MyMathLab offers several features that support adaptive learning: personalized homework and the adaptive study plan. These features allow your students to work on just what they need to learn when it makes the most sense, maximizing their potential for understanding and success.
- **Exercises:** The homework and practice exercises in MyMathLab are correlated to the exercises in the textbook, and they regenerate algorithmically to give students unlimited opportunity for practice and mastery. The software provides helpful feedback when students enter incorrect answers and includes optional learning aids including guided solutions, sample problems, animations, videos, and eText.
- **Learning and Teaching Tools** include:
    - **Learning Catalytics**—a "bring your own device" student engagement, assessment, and classroom intelligence system, included within MyMathLab. Includes questions written specifically for this course.
    - **Instructional videos**—full-length lecture videos as well as shorter example-based videos.
    - **MathTalk videos**—connect the math to the real world (particularly business). Also include assignable exercises to gauge student understanding of video content.
    - **Help for Gaps in Prerequisite Skills**—diagnostic quizzes tied to personalized assignments help address gaps in algebra skills that might otherwise impede success.
    - **Excel Spreadsheet Manual**—specifically written for this course.
    - **Graphing Calculator Manual**—specifically written for this course.
    - **Interactive Figures**—illustrate key concepts and allow manipulation for use as teaching and learning tools. Includes assignable exercises that require use of the figures.
- **Complete eText** is available to students through MyMathLab courses for the lifetime of the edition, giving students unlimited access to the eBook within any course using that edition of the textbook.
- **MyMathLab Accessibility:** MyMathLab is compatible with the JAWS screen reader, and enables multiple-choice and free-response problem types to be read and interacted with via keyboard controls and math notation input. MyMathLab also works with screen enlargers, including ZoomText, MAGic, and SuperNova. And all MyMathLab videos have closed captioning. More information on this functionality is available at http://mymathlab.com/accessibility.

MyMathLab comes from an **experienced partner** with educational expertise and an eye on the future.

- Knowing that you are using a Pearson product means knowing that you are using quality content. This means that our eTexts are accurate and our assessment tools work. It means we are committed to making MyMathLab as accessible as possible.
- Whether you are just getting started with MyMathLab, or have a question along the way, we're here to help you learn about our technologies and how to incorporate them into your course.
- To learn more about how MyMathLab combines proven learning applications with powerful assessment and continuously adaptive capabilities, visit www.mymathlab.com or contact your Pearson representative.

Additional Online Courseware Options (access code required)

- **MathXL®** is the homework and assessment engine that runs MyMathLab. (MyMathLab is MathXL plus a learning management system.)

- **MyMathLab Integrated Review Course** contains preassigned diagnostic and remediation resources for key prerequisite skills. Skills Check Quizzes help diagnose gaps in skills prior to each chapter. MyMathLab then provides personalized help on just those skills that a student has not mastered.
- **MyMathLab®Plus** contains all the features of MyMathLab with convenient management tools and a dedicated services team. It includes batch enrollment so everyone can be ready to start class on the first day, login directly from your campus portal, advanced reporting features, and 24/7 support by email or online chat.

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Raymond N. Greenwell
Nathan P. Ritchey

Prerequisite Skills Diagnostic Test

Below is a very brief test to help you recognize which, if any, prerequisite skills you may need to remediate in order to be successful in this course. After completing the test, check your answers in the back of the book. In addition to the answers, we have also provided the solutions to these problems in Appendix A. These solutions should help remind you how to solve the problems. For problems 5-26, the answers are followed by references to sections within Chapter R where you can find guidance on how to solve the problem and/or additional instruction. Addressing any weak prerequisite skills now will make a positive impact on your success as you progress through this course.

1. What percent of 50 is 10?

2. Simplify $\frac{13}{7} - \frac{2}{5}$.

3. Let x be the number of apples and y be the number of oranges. Write the following statement as an algebraic equation: "The total number of apples and oranges is 75."

4. Let s be the number of students and p be the number of professors. Write the following statement as an algebraic equation: "There are at least four times as many students as professors."

5. Solve for k: $7k + 8 = -4(3 - k)$.

6. Solve for x: $\frac{5}{8}x + \frac{1}{16}x = \frac{11}{16} + x$.

7. Write in interval notation: $-2 < x \leq 5$.

8. Using the variable x, write the following interval as an inequality: $(-\infty, -3]$.

9. Solve for y: $5(y - 2) + 1 \leq 7y + 8$.

10. Solve for p: $\frac{2}{3}(5p - 3) > \frac{3}{4}(2p + 1)$.

11. Carry out the operations and simplify: $(5y^2 - 6y - 4) - 2(3y^2 - 5y + 1)$.

12. Multiply out and simplify $(x^2 - 2x + 3)(x + 1)$.

13. Multiply out and simplify $(a - 2b)^2$.

14. Factor $3pq + 6p^2q + 9pq^2$.

15. Factor $3x^2 - x - 10$.

16. Perform the operation and simplify: $\frac{a^2 - 6a}{a^2 - 4} \cdot \frac{a - 2}{a}$.

17. Perform the operation and simplify: $\dfrac{x+3}{x^2-1} + \dfrac{2}{x^2+x}$.

18. Solve for x: $3x^2 + 4x = 1$.

19. Solve for z: $\dfrac{8z}{z+3} \leq 2$.

20. Simplify $\dfrac{4^{-1}(x^2y^3)^2}{x^{-2}y^5}$.

21. Simplify $\dfrac{4^{1/4}(p^{2/3}q^{-1/3})^{-1}}{4^{-1/4}p^{4/3}q^{4/3}}$.

22. Simplify as a single term without negative exponents: $k^{-1} - m^{-1}$.

23. Factor $(x^2+1)^{-1/2}(x+2) + 3(x^2+1)^{1/2}$.

24. Simplify $\sqrt[3]{64b^6}$.

25. Rationalize the denominator: $\dfrac{2}{4-\sqrt{10}}$.

26. Simplify $\sqrt{y^2 - 10y + 25}$.

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R Algebra Reference

In this chapter, we will review the most important topics in algebra. Knowing algebra is a fundamental prerequisite to success in higher mathematics. This algebra reference is designed for self-study; study it all at once or refer to it when needed throughout the course. Since this is a review, answers to all exercises are given in the answer section at the back of the book.

R.1 Polynomials

An expression such as $9p^4$ is a **term**; the number 9 is the **coefficient**, p is the **variable**, and 4 is the **exponent**. The expression p^4 means $p \cdot p \cdot p \cdot p$, while p^2 means $p \cdot p$, and so on. Terms having the same variable and the same exponent, such as $9x^4$ and $-3x^4$, are **like terms**. Terms that do not have both the same variable and the same exponent, such as m^2 and m^4, are **unlike terms**.

A **polynomial** is a term or a finite sum of terms in which all variables have whole number exponents, and no variables appear in denominators. Examples of polynomials include

$$5x^4 + 2x^3 + 6x, \qquad 8m^3 + 9m^2n - 6mn^2 + 3n^3, \qquad 10p, \qquad \text{and} \qquad -9.$$

Order of Operations

Algebra is a language, and you must be familiar with its rules to correctly interpret algebraic statements. The following order of operations has been agreed upon through centuries of usage.

- Expressions in **parentheses** (or other grouping symbols) are calculated first, working from the inside out. The numerator and denominator of a fraction are treated as expressions in parentheses.
- **Powers** are performed next, going from left to right.
- **Multiplication** and **division** are performed next, going from left to right.
- **Addition** and **subtraction** are performed last, going from left to right.

For example, in the expression $[6(x + 1)^2 + 3x - 22]^2$, suppose x has the value of 2. We would evaluate this as follows:

$$\begin{aligned}
[6(2 + 1)^2 + 3(2) - 22]^2 &= [6(3)^2 + 3(2) - 22]^2 && \text{Evaluate the expression in the innermost parentheses.}\\
&= [6(9) + 3(2) - 22]^2 && \text{Evaluate 3 raised to a power.}\\
&= (54 + 6 - 22)^2 && \text{Perform the multiplications.}\\
&= (38)^2 && \text{Perform the addition and subtraction from left to right.}\\
&= 1444 && \text{Evaluate the power.}
\end{aligned}$$

In the expression $\dfrac{x^2 + 3x + 6}{x + 6}$, suppose x has the value of 2. We would evaluate this as follows:

$$\begin{aligned}
\frac{2^2 + 3(2) + 6}{2 + 6} &= \frac{16}{8} && \text{Evaluate the numerator and the denominator.}\\
&= 2 && \text{Simplify the fraction.}
\end{aligned}$$

Adding and Subtracting Polynomials

The following properties of real numbers are useful for performing operations on polynomials.

Properties of Real Numbers

For all real numbers a, b, and c:

1. $a + b = b + a$; $ab = ba$;	**Commutative properties**
2. $(a + b) + c = a + (b + c)$; $(ab)c = a(bc)$;	**Associative properties**
3. $a(b + c) = ab + ac$.	**Distributive property**

EXAMPLE 1 Properties of Real Numbers

(a) $2 + x = x + 2$ — Commutative property of addition

(b) $x \cdot 3 = 3x$ — Commutative property of multiplication

(c) $(7x)x = 7(x \cdot x) = 7x^2$ — Associative property of multiplication

(d) $3(x + 4) = 3x + 12$ — Distributive property

One use of the distributive property is to add or subtract polynomials. Only like terms may be added or subtracted. For example,

$$12y^4 + 6y^4 = (12 + 6)y^4 = 18y^4,$$

and

$$-2m^2 + 8m^2 = (-2 + 8)m^2 = 6m^2,$$

but the polynomial $8y^4 + 2y^5$ cannot be further simplified. To subtract polynomials, we use the facts that $-(a + b) = -a - b$ and $-(a - b) = -a + b$. In the next example, we show how to add and subtract polynomials.

EXAMPLE 2 Adding and Subtracting Polynomials

Add or subtract as indicated.

(a) $(8x^3 - 4x^2 + 6x) + (3x^3 + 5x^2 - 9x + 8)$

SOLUTION Combine like terms.

$$\begin{aligned}&(8x^3 - 4x^2 + 6x) + (3x^3 + 5x^2 - 9x + 8)\\&= (8x^3 + 3x^3) + (-4x^2 + 5x^2) + (6x - 9x) + 8\\&= 11x^3 + x^2 - 3x + 8\end{aligned}$$

(b) $2(-4x^4 + 6x^3 - 9x^2 - 12) + 3(-3x^3 + 8x^2 - 11x + 7)$

SOLUTION Multiply each polynomial by the factor in front of the polynomial, and then combine terms as before.

$$\begin{aligned}&2(-4x^4 + 6x^3 - 9x^2 - 12) + 3(-3x^3 + 8x^2 - 11x + 7)\\&= -8x^4 + 12x^3 - 18x^2 - 24 - 9x^3 + 24x^2 - 33x + 21\\&= -8x^4 + 3x^3 + 6x^2 - 33x - 3\end{aligned}$$

(c) $(2x^2 - 11x + 8) - (7x^2 - 6x + 2)$

SOLUTION Distributing the minus sign and combining like terms yields

$$\begin{aligned}&(2x^2 - 11x + 8) + (-7x^2 + 6x - 2)\\&= -5x^2 - 5x + 6.\end{aligned}$$

TRY YOUR TURN 1

YOUR TURN 1 Perform the operation $3(x^2 - 4x - 5) - 4(3x^2 - 5x - 7)$.

Multiplying Polynomials

The distributive property is also used to multiply polynomials, along with the fact that $a^m \cdot a^n = a^{m+n}$. For example,

$$x \cdot x = x^1 \cdot x^1 = x^{1+1} = x^2 \quad \text{and} \quad x^2 \cdot x^5 = x^{2+5} = x^7.$$

EXAMPLE 3 Multiplying Polynomials

Multiply.

(a) $8x(6x - 4)$

SOLUTION Using the distributive property yields

$$\begin{aligned}8x(6x - 4) &= 8x(6x) - 8x(4)\\&= 48x^2 - 32x.\end{aligned}$$

(b) $(3p - 2)(p^2 + 5p - 1)$

SOLUTION Using the distributive property yields

$$\begin{aligned}(3p - 2)&(p^2 + 5p - 1)\\ &= 3p(p^2 + 5p - 1) - 2(p^2 + 5p - 1)\\ &= 3p(p^2) + 3p(5p) + 3p(-1) - 2(p^2) - 2(5p) - 2(-1)\\ &= 3p^3 + 15p^2 - 3p - 2p^2 - 10p + 2\\ &= 3p^3 + 13p^2 - 13p + 2.\end{aligned}$$

(c) $(x + 2)(x + 3)(x - 4)$

SOLUTION Multiplying the first two polynomials and then multiplying their product by the third polynomial yields

$$\begin{aligned}(x + 2)&(x + 3)(x - 4)\\ &= [(x + 2)(x + 3)](x - 4)\\ &= (x^2 + 2x + 3x + 6)(x - 4)\\ &= (x^2 + 5x + 6)(x - 4)\\ &= x^3 - 4x^2 + 5x^2 - 20x + 6x - 24\\ &= x^3 + x^2 - 14x - 24.\end{aligned}$$

TRY YOUR TURN 2

YOUR TURN 2 Perform the operation $(3y + 2)(4y^2 - 2y - 5)$.

A **binomial** is a polynomial with exactly two terms, such as $2x + 1$ or $m + n$. When two binomials are multiplied, the FOIL method (First, Outer, Inner, Last) is used as a memory aid.

EXAMPLE 4 Multiplying Polynomials

Find $(2m - 5)(m + 4)$ using the FOIL method.

SOLUTION

$$\begin{aligned}(2m - 5)(m + 4) &= \overset{\text{F}}{(2m)(m)} + \overset{\text{O}}{(2m)(4)} + \overset{\text{I}}{(-5)(m)} + \overset{\text{L}}{(-5)(4)}\\ &= 2m^2 + 8m - 5m - 20\\ &= 2m^2 + 3m - 20\end{aligned}$$

TRY YOUR TURN 3

YOUR TURN 3 Find $(2x + 7)(3x - 1)$ using the FOIL method.

EXAMPLE 5 Multiplying Polynomials

Find $(2k - 5m)^3$.

SOLUTION Write $(2k - 5m)^3$ as $(2k - 5m)(2k - 5m)(2k - 5m)$. Then multiply the first two factors using FOIL.

$$\begin{aligned}(2k - 5m)(2k - 5m) &= 4k^2 - 10km - 10km + 25m^2\\ &= 4k^2 - 20km + 25m^2\end{aligned}$$

Now multiply this last result by $(2k - 5m)$ using the distributive property, as in Example 3(c).

$$\begin{aligned}(4k^2 &- 20km + 25m^2)(2k - 5m)\\ &= 4k^2(2k - 5m) - 20km(2k - 5m) + 25m^2(2k - 5m)\\ &= 8k^3 - 20k^2m - 40k^2m + 100km^2 + 50km^2 - 125m^3\\ &= 8k^3 - 60k^2m + 150km^2 - 125m^3 \qquad \text{Combine like terms.}\end{aligned}$$

TRY YOUR TURN 4

YOUR TURN 4 Find $(3x + 2y)^3$.

Notice in the first part of Example 5, when we multiplied $(2k - 5m)$ by itself, that the product of the square of a binomial is the square of the first term, $(2k)^2$, plus twice the product of the two terms, $(2)(2k)(-5m)$, plus the square of the last term, $(-5k)^2$.

CAUTION Avoid the common error of writing $(x + y)^2 = x^2 + y^2$. As the first step of Example 5 shows, the square of a binomial has three terms, so

$$(x + y)^2 = x^2 + 2xy + y^2.$$

Furthermore, higher powers of a binomial also result in more than two terms. For example, verify by multiplication that

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

Remember, for any value of $n \neq 1$,

$$(x + y)^n \neq x^n + y^n.$$

R.1 EXERCISES

Perform the indicated operations.

1. $(2x^2 - 6x + 11) + (-3x^2 + 7x - 2)$

2. $(-4y^2 - 3y + 8) - (2y^2 - 6y - 2)$

3. $-6(2q^2 + 4q - 3) + 4(-q^2 + 7q - 3)$

4. $2(3r^2 + 4r + 2) - 3(-r^2 + 4r - 5)$

5. $(0.613x^2 - 4.215x + 0.892) - 0.47(2x^2 - 3x + 5)$

6. $0.5(5r^2 + 3.2r - 6) - (1.7r^2 - 2r - 1.5)$

7. $-9m(2m^2 + 3m - 1)$

8. $6x(-2x^3 + 5x + 6)$

9. $(3t - 2y)(3t + 5y)$

10. $(9k + q)(2k - q)$

11. $(2 - 3x)(2 + 3x)$

12. $(6m + 5)(6m - 5)$

13. $\left(\frac{2}{5}y + \frac{1}{8}z\right)\left(\frac{3}{5}y + \frac{1}{2}z\right)$

14. $\left(\frac{3}{4}r - \frac{2}{3}s\right)\left(\frac{5}{4}r + \frac{1}{3}s\right)$

15. $(3p - 1)(9p^2 + 3p + 1)$

16. $(3p + 2)(5p^2 + p - 4)$

17. $(2m + 1)(4m^2 - 2m + 1)$

18. $(k + 2)(12k^3 - 3k^2 + k + 1)$

19. $(x + y + z)(3x - 2y - z)$

20. $(r + 2s - 3t)(2r - 2s + t)$

21. $(x + 1)(x + 2)(x + 3)$

22. $(x - 1)(x + 2)(x - 3)$

23. $(x + 2)^2$

24. $(2a - 4b)^2$

25. $(x - 2y)^3$

26. $(3x + y)^3$

YOUR TURN ANSWERS

1. $-9x^2 + 8x + 13$ **2.** $12y^3 + 2y^2 - 19y - 10$

3. $6x^2 + 19x - 7$

4. $27x^3 + 54x^2y + 36xy^2 + 8y^3$

R.2 Factoring

Multiplication of polynomials relies on the distributive property. The reverse process, where a polynomial is written as a product of other polynomials, is called **factoring**. For example, one way to factor the number 18 is to write it as the product $9 \cdot 2$; both 9 and 2 are **factors** of 18. Usually, only integers are used as factors of integers. The number 18 can also be written with three integer factors as $2 \cdot 3 \cdot 3$.

The Greatest Common Factor To factor the algebraic expression $15m + 45$, first note that both $15m$ and 45 are divisible by 15; $15m = 15 \cdot m$ and $45 = 15 \cdot 3$. By the distributive property,

$$15m + 45 = 15 \cdot m + 15 \cdot 3 = 15(m + 3).$$

Both 15 and $m + 3$ are factors of $15m + 45$. Since 15 divides into both terms of $15m + 45$ (and is the largest number that will do so), 15 is the **greatest common factor** for the polynomial $15m + 45$. The process of writing $15m + 45$ as $15(m + 3)$ is often called **factoring out** the greatest common factor.

EXAMPLE 1 Factoring

Factor out the greatest common factor.

(a) $12p - 18q$

SOLUTION Both $12p$ and $18q$ are divisible by 6. Therefore,

$$12p - 18q = 6 \cdot 2p - 6 \cdot 3q = 6(2p - 3q).$$

(b) $8x^3 - 9x^2 + 15x$

SOLUTION Each of these terms is divisible by x.

$$8x^3 - 9x^2 + 15x = (8x^2) \cdot x - (9x) \cdot x + 15 \cdot x$$
$$= x(8x^2 - 9x + 15) \quad \text{or} \quad (8x^2 - 9x + 15)x$$

TRY YOUR TURN 1

YOUR TURN 1 Factor $4z^4 + 4z^3 + 18z^2$.

One can always check factorization by finding the product of the factors and comparing it to the original expression.

CAUTION When factoring out the greatest common factor in an expression like $2x^2 + x$, be careful to remember the 1 in the second term.

$$2x^2 + x = 2x^2 + 1x = x(2x + 1), \qquad \text{not } x(2x).$$

Factoring Trinomials A polynomial that has no greatest common factor (other than 1) may still be factorable. For example, the polynomial $x^2 + 5x + 6$ can be factored as $(x + 2)(x + 3)$. To see that this is correct, find the product $(x + 2)(x + 3)$; you should get $x^2 + 5x + 6$. A polynomial such as this with three terms is called a **trinomial**. To factor a trinomial of the form $x^2 + bx + c$, where the coefficient of x^2 is 1, use FOIL backwards. We look for two factors of c whose sum is b. When c is positive, its factors must have the same sign. Since b is the sum of these two factors, the factors will have the same sign as b. When c is negative, its factors have opposite signs. Again, since b is the sum of these two factors, the factor with the greater absolute value will have the same sign as b.

EXAMPLE 2 Factoring a Trinomial

Factor $y^2 + 8y + 15$.

SOLUTION Since the coefficient of y^2 is 1, factor by finding two numbers whose *product* is 15 and whose *sum* is 8. Because the constant and the middle term are positive, the numbers must both be positive. Begin by listing all pairs of positive integers having a product of 15. As you do this, also form the sum of each pair of numbers.

Products	Sums
$15 \cdot 1 = 15$	$15 + 1 = 16$
$5 \cdot 3 = 15$	$5 + 3 = 8$

The numbers 5 and 3 have a product of 15 and a sum of 8. Thus, $y^2 + 8y + 15$ factors as

$$y^2 + 8y + 15 = (y + 5)(y + 3).$$

The answer can also be written as $(y + 3)(y + 5)$. TRY YOUR TURN 2

YOUR TURN 2 Factor $x^2 - 3x - 10$.

If the coefficient of the squared term is *not* 1, work as shown on the next page.

EXAMPLE 3 Factoring a Trinomial

Factor $4x^2 + 8xy - 5y^2$.

SOLUTION The possible factors of $4x^2$ are $4x$ and x or $2x$ and $2x$; the possible factors of $-5y^2$ are $-5y$ and y or $5y$ and $-y$. Try various combinations of these factors until one works (if, indeed, any work). For example, try the product $(x + 5y)(4x - y)$.

$$\begin{aligned}(x + 5y)(4x - y) &= 4x^2 - xy + 20xy - 5y^2 \\ &= 4x^2 + 19xy - 5y^2\end{aligned}$$

This product is not correct, so try another combination.

$$\begin{aligned}(2x - y)(2x + 5y) &= 4x^2 + 10xy - 2xy - 5y^2 \\ &= 4x^2 + 8xy - 5y^2\end{aligned}$$

YOUR TURN 3 Factor $6a^2 + 5ab - 4b^2$.

Since this combination gives the correct polynomial,

$$4x^2 + 8xy - 5y^2 = (2x - y)(2x + 5y).$$

TRY YOUR TURN 3

Special Factorizations

Four special factorizations occur so often that they are listed here for future reference.

Special Factorizations

$x^2 - y^2 = (x + y)(x - y)$	**Difference of two squares**
$x^2 + 2xy + y^2 = (x + y)^2$	**Perfect square**
$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$	**Difference of two cubes**
$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$	**Sum of two cubes**

A polynomial that cannot be factored is called a **prime polynomial**.

EXAMPLE 4 Factoring Polynomials

Factor each polynomial, if possible.

(a) $64p^2 - 49q^2 = (8p)^2 - (7q)^2 = (8p + 7q)(8p - 7q)$ — Difference of two squares

(b) $x^2 + 36$ is a prime polynomial.

(c) $x^2 + 12x + 36 = x^2 + 2(x)(6) + 6^2 = (x + 6)^2$ — Perfect square

(d) $9y^2 - 24yz + 16z^2 = (3y)^2 + 2(3y)(-4z) + (-4z)^2$ — Perfect square

$$= [3y + (-4z)]^2 = (3y - 4z)^2$$

(e) $y^3 - 8 = y^3 - 2^3 = (y - 2)(y^2 + 2y + 4)$ — Difference of two cubes

(f) $m^3 + 125 = m^3 + 5^3 = (m + 5)(m^2 - 5m + 25)$ — Sum of two cubes

(g) $8k^3 - 27z^3 = (2k)^3 - (3z)^3 = (2k - 3z)(4k^2 + 6kz + 9z^2)$ — Difference of two cubes

(h) $p^4 - 1 = (p^2 + 1)(p^2 - 1) = (p^2 + 1)(p + 1)(p - 1)$ — Difference of two squares

CAUTION In factoring, always look for a common factor first. Since $36x^2 - 4y^2$ has a common factor of 4,

$$36x^2 - 4y^2 = 4(9x^2 - y^2) = 4(3x + y)(3x - y).$$

It would be incomplete to factor it as

$$36x^2 - 4y^2 = (6x + 2y)(6x - 2y),$$

since each factor can be factored still further. To *factor* means to factor completely, so that each polynomial factor is prime.

R.2 EXERCISES

Factor each polynomial. If a polynomial cannot be factored, write *prime*. Factor out the greatest common factor as necessary.

1. $7a^3 + 14a^2$

2. $3y^3 + 24y^2 + 9y$

3. $13p^4q^2 - 39p^3q + 26p^2q^2$

4. $60m^4 - 120m^3n + 50m^2n^2$

5. $m^2 - 5m - 14$

6. $x^2 + 4x - 5$

7. $z^2 + 9z + 20$

8. $b^2 - 8b + 7$

9. $a^2 - 6ab + 5b^2$

10. $s^2 + 2st - 35t^2$

11. $y^2 - 4yz - 21z^2$

12. $3x^2 + 4x - 7$

13. $3a^2 + 10a + 7$

14. $15y^2 + y - 2$

15. $21m^2 + 13mn + 2n^2$

16. $6a^2 - 48a - 120$

17. $3m^3 + 12m^2 + 9m$

18. $4a^2 + 10a + 6$

19. $24a^4 + 10a^3b - 4a^2b^2$

20. $24x^4 + 36x^3y - 60x^2y^2$

21. $x^2 - 64$

22. $9m^2 - 25$

23. $10x^2 - 160$

24. $9x^2 + 64$

25. $z^2 + 14zy + 49y^2$

26. $s^2 - 10st + 25t^2$

27. $9p^2 - 24p + 16$

28. $a^3 - 216$

29. $27r^3 - 64s^3$

30. $3m^3 + 375$

31. $x^4 - y^4$

32. $16a^4 - 81b^4$

YOUR TURN ANSWERS

1. $2z^2(2z^2 + 2z + 9)$

2. $(x + 2)(x - 5)$

3. $(2a - b)(3a + 4b)$

R.3 Rational Expressions

Many algebraic fractions are **rational expressions**, which are quotients of polynomials with nonzero denominators. Examples include

$$\frac{8}{x-1}, \quad \frac{3x^2+4x}{5x-6}, \quad \text{and} \quad \frac{2y+1}{y^2}.$$

Next, we summarize properties for working with rational expressions.

Properties of Rational Expressions

For all mathematical expressions P, Q, R, and S, with $Q \neq 0$ and $S \neq 0$:

$\frac{P}{Q} = \frac{PS}{QS}$	**Fundamental property**
$\frac{P}{Q} + \frac{R}{Q} = \frac{P+R}{Q}$	**Addition**
$\frac{P}{Q} - \frac{R}{Q} = \frac{P-R}{Q}$	**Subtraction**
$\frac{P}{Q} \cdot \frac{R}{S} = \frac{PR}{QS}$	**Multiplication**
$\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \cdot \frac{S}{R} \quad (R \neq 0)$	**Division**

When writing a rational expression in lowest terms, we may need to use the fact that $\frac{a^m}{a^n} = a^{m-n}$. For example,

$$\frac{x^4}{3x} = \frac{1x^4}{3x} = \frac{1}{3} \cdot \frac{x^4}{x} = \frac{1}{3} \cdot x^{4-1} = \frac{1}{3}x^3 = \frac{x^3}{3}.$$

EXAMPLE 1 Reducing Rational Expressions

Write each rational expression in lowest terms, that is, reduce the expression as much as possible.

(a) $$\frac{8x+16}{4} = \frac{8(x+2)}{4} = \frac{4 \cdot 2(x+2)}{4} = 2(x+2)$$

Factor both the numerator and denominator in order to identify any common factors, which have a quotient of 1. The answer could also be written as $2x + 4$.

(b) $$\frac{k^2+7k+12}{k^2+2k-3} = \frac{(k+4)(k+3)}{(k-1)(k+3)} = \frac{k+4}{k-1}$$

The answer cannot be further reduced.

TRY YOUR TURN 1

YOUR TURN 1 Write in lowest terms $\frac{z^2+5z+6}{2z^2+7z+3}$.

CAUTION One of the most common errors in algebra involves incorrect use of the fundamental property of rational expressions. Only common *factors* may be divided or "canceled." It is essential to factor rational expressions before writing them in lowest terms. In Example 1(b), for instance, it is not correct to "cancel" k^2 (or cancel k, or divide 12 by -3) because the additions and subtraction must be performed first. Here they cannot be performed, so it is not possible to divide. After factoring, however, the fundamental property can be used to write the expression in lowest terms.

EXAMPLE 2 Combining Rational Expressions

Perform each operation.

(a) $$\frac{3y+9}{6} \cdot \frac{18}{5y+15}$$

SOLUTION Factor where possible, then multiply numerators and denominators and reduce to lowest terms.

$$\frac{3y+9}{6} \cdot \frac{18}{5y+15} = \frac{3(y+3)}{6} \cdot \frac{18}{5(y+3)}$$ Factor.

$$= \frac{3 \cdot 18(y+3)}{6 \cdot 5(y+3)}$$ Multiply.

$$= \frac{3 \cdot 6 \cdot 3(y+3)}{6 \cdot 5(y+3)} = \frac{3 \cdot 3}{5} = \frac{9}{5}$$ Reduce to lowest terms.

(b) $$\frac{m^2+5m+6}{m+3} \cdot \frac{m}{m^2+3m+2}$$

SOLUTION Factor where possible.

$$\frac{(m+2)(m+3)}{m+3} \cdot \frac{m}{(m+2)(m+1)}$$ Factor.

$$= \frac{m(m+2)(m+3)}{(m+3)(m+2)(m+1)} = \frac{m}{m+1}$$ Reduce to lowest terms.

(c) $$\frac{9p-36}{12} \div \frac{5(p-4)}{18}$$

SOLUTION Use the division property of rational expressions.

$$\frac{9p-36}{12} \div \frac{5(p-4)}{18} = \frac{9p-36}{12} \cdot \frac{18}{5(p-4)}$$ Invert and multiply.

$$= \frac{9(p-4)}{6 \cdot 2} \cdot \frac{6 \cdot 3}{5(p-4)} = \frac{27}{10}$$ Factor and reduce to lowest terms.

(d) $\dfrac{4}{5k} - \dfrac{11}{5k}$

SOLUTION As shown in the list of properties, to subtract two rational expressions that have the same denominators, subtract the numerators while keeping the same denominator.

$$\frac{4}{5k} - \frac{11}{5k} = \frac{4 - 11}{5k} = -\frac{7}{5k}$$

(e) $\dfrac{7}{p} + \dfrac{9}{2p} + \dfrac{1}{3p}$

SOLUTION These three fractions cannot be added until their denominators are the same. A **common denominator** into which p, $2p$, and $3p$ all divide is $6p$. Note that $12p$ is also a common denominator, but $6p$ is the **least common denominator**. Use the fundamental property to rewrite each rational expression with a denominator of $6p$.

$$\frac{7}{p} + \frac{9}{2p} + \frac{1}{3p} = \frac{6 \cdot 7}{6 \cdot p} + \frac{3 \cdot 9}{3 \cdot 2p} + \frac{2 \cdot 1}{2 \cdot 3p}$$ Rewrite with common denominator $6p$.

$$= \frac{42}{6p} + \frac{27}{6p} + \frac{2}{6p}$$

$$= \frac{42 + 27 + 2}{6p}$$

$$= \frac{71}{6p}$$

(f) $\dfrac{x + 1}{x^2 + 5x + 6} - \dfrac{5x - 1}{x^2 - x - 12}$

SOLUTION To find the least common denominator, we first factor each denominator. Then we change each fraction so they all have the same denominator, being careful to multiply only by quotients that equal 1.

$$\frac{x + 1}{x^2 + 5x + 6} - \frac{5x - 1}{x^2 - x - 12}$$

$$= \frac{x + 1}{(x + 2)(x + 3)} - \frac{5x - 1}{(x + 3)(x - 4)}$$ Factor denominators.

$$= \frac{x + 1}{(x + 2)(x + 3)} \cdot \frac{(x - 4)}{(x - 4)} - \frac{5x - 1}{(x + 3)(x - 4)} \cdot \frac{(x + 2)}{(x + 2)}$$ Rewrite with common denominators.

$$= \frac{(x^2 - 3x - 4) - (5x^2 + 9x - 2)}{(x + 2)(x + 3)(x - 4)}$$ Multiply numerators.

$$= \frac{-4x^2 - 12x - 2}{(x + 2)(x + 3)(x - 4)}$$ Subtract.

$$= \frac{-2(2x^2 + 6x + 1)}{(x + 2)(x + 3)(x - 4)}$$ Factor numerator.

Because the numerator cannot be factored further, we leave our answer in this form. We could also multiply out the denominator, but factored form is usually more useful.

TRY YOUR TURN 2

YOUR TURN 2 Perform each of the following operations.

(a) $\dfrac{z^2 + 5z + 6}{2z^2 - 5z - 3} \cdot \dfrac{2z^2 - z - 1}{z^2 + 2z - 3}$.

(b) $\dfrac{a - 3}{a^2 + 3a + 2} + \dfrac{5a}{a^2 - 4}$.

R.3 EXERCISES

Write each rational expression in lowest terms.

1. $\dfrac{5v^2}{35v}$

2. $\dfrac{25p^3}{10p^2}$

3. $\dfrac{8k+16}{9k+18}$

4. $\dfrac{2(t-15)}{(t-15)(t+2)}$

5. $\dfrac{4x^3-8x^2}{4x^2}$

6. $\dfrac{36y^2+72y}{9y}$

7. $\dfrac{m^2-4m+4}{m^2+m-6}$

8. $\dfrac{r^2-r-6}{r^2+r-12}$

9. $\dfrac{3x^2+3x-6}{x^2-4}$

10. $\dfrac{z^2-5z+6}{z^2-4}$

11. $\dfrac{m^4-16}{4m^2-16}$

12. $\dfrac{6y^2+11y+4}{3y^2+7y+4}$

Perform the indicated operations.

13. $\dfrac{9k^2}{25}\cdot\dfrac{5}{3k}$

14. $\dfrac{15p^3}{9p^2}\div\dfrac{6p}{10p^2}$

15. $\dfrac{3a+3b}{4c}\cdot\dfrac{12}{5(a+b)}$

16. $\dfrac{a-3}{16}\div\dfrac{a-3}{32}$

17. $\dfrac{2k-16}{6}\div\dfrac{4k-32}{3}$

18. $\dfrac{9y-18}{6y+12}\cdot\dfrac{3y+6}{15y-30}$

19. $\dfrac{4a+12}{2a-10}\div\dfrac{a^2-9}{a^2-a-20}$

20. $\dfrac{6r-18}{9r^2+6r-24}\cdot\dfrac{12r-16}{4r-12}$

21. $\dfrac{k^2+4k-12}{k^2+10k+24}\cdot\dfrac{k^2+k-12}{k^2-9}$

22. $\dfrac{m^2+3m+2}{m^2+5m+4}\div\dfrac{m^2+5m+6}{m^2+10m+24}$

23. $\dfrac{2m^2-5m-12}{m^2-10m+24}\div\dfrac{4m^2-9}{m^2-9m+18}$

24. $\dfrac{4n^2+4n-3}{6n^2-n-15}\cdot\dfrac{8n^2+32n+30}{4n^2+16n+15}$

25. $\dfrac{a+1}{2}-\dfrac{a-1}{2}$

26. $\dfrac{3}{p}+\dfrac{1}{2}$

27. $\dfrac{6}{5y}-\dfrac{3}{2}$

28. $\dfrac{1}{6m}+\dfrac{2}{5m}+\dfrac{4}{m}$

29. $\dfrac{1}{m-1}+\dfrac{2}{m}$

30. $\dfrac{5}{2r+3}-\dfrac{2}{r}$

31. $\dfrac{8}{3(a-1)}+\dfrac{2}{a-1}$

32. $\dfrac{2}{5(k-2)}+\dfrac{3}{4(k-2)}$

33. $\dfrac{4}{x^2+4x+3}+\dfrac{3}{x^2-x-2}$

34. $\dfrac{y}{y^2+2y-3}-\dfrac{1}{y^2+4y+3}$

35. $\dfrac{3k}{2k^2+3k-2}-\dfrac{2k}{2k^2-7k+3}$

36. $\dfrac{4m}{3m^2+7m-6}-\dfrac{m}{3m^2-14m+8}$

37. $\dfrac{2}{a+2}+\dfrac{1}{a}+\dfrac{a-1}{a^2+2a}$

38. $\dfrac{5x+2}{x^2-1}+\dfrac{3}{x^2+x}-\dfrac{1}{x^2-x}$

YOUR TURN ANSWERS

1. $(z+2)/(2z+1)$

2. **(a)** $(z+2)/(z-3)$

(b) $6(a^2+1)/[(a-2)(a+2)(a+1)]$

R.4 Equations

Linear Equations

Equations that can be written in the form $ax+b=0$, where a and b are real numbers, with $a \neq 0$, are **linear equations**. Examples of linear equations include $5y+9=16$, $8x=4$, and $-3p+5=-8$. Equations that are *not* linear include absolute value equations such as $|x|=4$. The following properties are used to solve linear equations.

Properties of Equality

For all real numbers a, b, and c:

1. **If $a=b$, then $a+c=b+c$.** (The same number may be added to both sides of an equation.) — **Addition property of equality**

2. **If $a=b$, then $ac=bc$.** (Both sides of an equation may be multiplied by the same number.) — **Multiplication property of equality**

EXAMPLE 1 Solving Linear Equations

Solve the following equations.

(a) $x - 2 = 3$

SOLUTION The goal is to isolate the variable. Using the addition property of equality yields

$$x - 2 + 2 = 3 + 2, \qquad \text{or} \qquad x = 5.$$

(b) $\frac{x}{2} = 3$

SOLUTION Using the multiplication property of equality yields

$$2 \cdot \frac{x}{2} = 2 \cdot 3, \qquad \text{or} \qquad x = 6.$$

The following example shows how these properties are used to solve linear equations. The goal is to isolate the variable. The solutions should always be checked by substitution into the original equation.

EXAMPLE 2 Solving a Linear Equation

Solve $2x - 5 + 8 = 3x + 2(2 - 3x)$.

SOLUTION

$$2x - 5 + 8 = 3x + 4 - 6x \qquad \text{Distributive property}$$
$$2x + 3 = -3x + 4 \qquad \text{Combine like terms.}$$
$$5x + 3 = 4 \qquad \text{Add } 3x \text{ to both sides.}$$
$$5x = 1 \qquad \text{Add } -3 \text{ to both sides.}$$
$$x = \frac{1}{5} \qquad \text{Multiply both sides by } \tfrac{1}{5}.$$

YOUR TURN 1 Solve $3x - 7 = 4(5x + 2) - 7x$.

Check by substituting into the original equation. The left side becomes $2(1/5) - 5 + 8$ and the right side becomes $3(1/5) + 2[2 - 3(1/5)]$. Verify that both of these expressions simplify to $17/5$.

TRY YOUR TURN 1

Quadratic Equations

An equation with 2 as the greatest exponent of the variable is a *quadratic equation*. A **quadratic equation** has the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$. A quadratic equation written in the form $ax^2 + bx + c = 0$ is said to be in **standard form**.

The simplest way to solve a quadratic equation, but one that is not always applicable, is by factoring. This method depends on the **zero-factor property**.

Zero-Factor Property

If a and b are real numbers, with $ab = 0$, then either

$$a = 0 \quad \text{or} \quad b = 0 \quad \text{(or both).}$$

EXAMPLE 3 Solving a Quadratic Equation

Solve $6r^2 + 7r = 3$.

SOLUTION First write the equation in standard form.

$$6r^2 + 7r - 3 = 0$$

Now factor $6r^2 + 7r - 3$ to get

$$(3r - 1)(2r + 3) = 0.$$

By the zero-factor property, the product $(3r - 1)(2r + 3)$ can equal 0 if and only if

$$3r - 1 = 0 \quad \text{or} \quad 2r + 3 = 0.$$

YOUR TURN 2 Solve $2m^2 + 7m = 15$.

Solve each of these equations separately to find that the solutions are $1/3$ and $-3/2$. Check these solutions by substituting them into the original equation. **TRY YOUR TURN 2**

CAUTION Remember, the zero-factor property requires that the product of two (or more) factors be equal to *zero*, not some other quantity. It would be incorrect to use the zero-factor property with an equation in the form $(x + 3)(x - 1) = 4$, for example.

If a quadratic equation cannot be solved easily by factoring, use the *quadratic formula.* (The derivation of the quadratic formula is given in most algebra books.)

Quadratic Formula

The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

EXAMPLE 4 Quadratic Formula

Solve $x^2 - 4x - 5 = 0$ by the quadratic formula.

SOLUTION The equation is already in standard form (it has 0 alone on one side of the equal sign), so the values of a, b, and c from the quadratic formula are easily identified. The coefficient of the squared term gives the value of a; here, $a = 1$. Also, $b = -4$ and $c = -5$, where b is the coefficient of the linear term and c is the constant coefficient. (Be careful to use the correct signs.) Substitute these values into the quadratic formula.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-5)}}{2(1)} \qquad a = 1, b = -4, c = -5$$

$$x = \frac{4 \pm \sqrt{16 + 20}}{2} \qquad (-4)^2 = (-4)(-4) = 16$$

$$x = \frac{4 \pm 6}{2} \qquad \sqrt{16 + 20} = \sqrt{36} = 6$$

The $\pm$ sign represents the two solutions of the equation. To find both of the solutions, first use + and then use −.

$$x = \frac{4 + 6}{2} = \frac{10}{2} = 5 \quad \text{or} \quad x = \frac{4 - 6}{2} = \frac{-2}{2} = -1$$

The two solutions are 5 and −1.

CAUTION Notice in the quadratic formula that the square root is added to or subtracted from the value of $-b$ *before* dividing by $2a$.

EXAMPLE 5 Quadratic Formula

Solve $x^2 + 1 = 4x$.

SOLUTION First, add $-4x$ on both sides of the equal sign in order to get the equation in standard form.

$$x^2 - 4x + 1 = 0$$

Now identify the values of a, b, and c. Here $a = 1$, $b = -4$, and $c = 1$. Substitute these numbers into the quadratic formula.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)} \qquad a = 1, b = -4, c = 1$$

$$= \frac{4 \pm \sqrt{16 - 4}}{2}$$

$$= \frac{4 \pm \sqrt{12}}{2}$$

Simplify the solutions by writing $\sqrt{12}$ as $\sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}$. Substituting $2\sqrt{3}$ for $\sqrt{12}$ gives

$$x = \frac{4 \pm 2\sqrt{3}}{2}$$

$$= \frac{2(2 \pm \sqrt{3})}{2} \qquad \text{Factor } 4 \pm 2\sqrt{3}.$$

$$= 2 \pm \sqrt{3}. \qquad \text{Reduce to lowest terms.}$$

The two solutions are $2 + \sqrt{3}$ and $2 - \sqrt{3}$.

The exact values of the solutions are $2 + \sqrt{3}$ and $2 - \sqrt{3}$. The $\sqrt{\ }$ key on a calculator gives decimal approximations of these solutions (to the nearest thousandth):

YOUR TURN 3 Solve $z^2 + 6 = 8z$.

$$2 + \sqrt{3} \approx 2 + 1.732 = 3.732^*$$

$$2 - \sqrt{3} \approx 2 - 1.732 = 0.268$$

TRY YOUR TURN 3

NOTE Sometimes the quadratic formula will give a result with a negative number under the radical sign, such as $3 \pm \sqrt{-5}$. A solution of this type is a complex number. Since this text deals only with real numbers, such solutions cannot be used.

Equations with Fractions

When an equation includes fractions, first eliminate all denominators by multiplying both sides of the equation by a common denominator, a number that can be divided (with no remainder) by each denominator in the equation. When an equation involves fractions with variable denominators, it is *necessary* to check all solutions in the original equation to be sure that no solution will lead to a zero denominator.

EXAMPLE 6 Solving Rational Equations

Solve each equation.

(a) $\frac{r}{10} - \frac{2}{15} = \frac{3r}{20} - \frac{1}{5}$

SOLUTION The denominators are 10, 15, 20, and 5. Each of these numbers can be divided into 60, so 60 is a common denominator. Multiply both sides of the equation by

*The symbol $\approx$ means "is approximately equal to."

60 and use the distributive property. (If a common denominator cannot be found easily, all the denominators in the problem can be multiplied together to produce one.)

$$\frac{r}{10} - \frac{2}{15} = \frac{3r}{20} - \frac{1}{5}$$

$$60\left(\frac{r}{10} - \frac{2}{15}\right) = 60\left(\frac{3r}{20} - \frac{1}{5}\right) \quad \text{Multiply by the common denominator.}$$

$$60\left(\frac{r}{10}\right) - 60\left(\frac{2}{15}\right) = 60\left(\frac{3r}{20}\right) - 60\left(\frac{1}{5}\right) \quad \text{Distributive property}$$

$$6r - 8 = 9r - 12$$

Add $-9r$ and 8 to both sides.

$$6r - 8 + (-9r) + 8 = 9r - 12 + (-9r) + 8$$

$$-3r = -4$$

$$r = \frac{4}{3} \quad \text{Multiply each side by } -\tfrac{1}{3}.$$

Check by substituting into the original equation.

(b) $\dfrac{3}{x^2} - 12 = 0$

SOLUTION Begin by multiplying both sides of the equation by x^2 to get $3 - 12x^2 = 0$. This equation could be solved by using the quadratic formula with $a = -12$, $b = 0$, and $c = 3$. Another method that works well for the type of quadratic equation in which $b = 0$ is shown below.

$$3 - 12x^2 = 0$$

$$3 = 12x^2 \quad \text{Add } 12x^2.$$

$$\frac{1}{4} = x^2 \quad \text{Multiply by } \tfrac{1}{12}.$$

$$\pm\frac{1}{2} = x \quad \text{Take square roots.}$$

Verify that there are two solutions, $-1/2$ and $1/2$.

(c) $\dfrac{2}{k} - \dfrac{3k}{k+2} = \dfrac{k}{k^2 + 2k}$

SOLUTION Factor $k^2 + 2k$ as $k(k + 2)$. The least common denominator for all the fractions is $k(k + 2)$. Multiplying both sides by $k(k + 2)$ gives the following:

$$k(k+2)\cdot\left(\frac{2}{k} - \frac{3k}{k+2}\right) = k(k+2)\cdot\frac{k}{k^2 + 2k}$$

$$2(k + 2) - 3k(k) = k$$

$$2k + 4 - 3k^2 = k \quad \text{Distributive property}$$

$$-3k^2 + k + 4 = 0 \quad \text{Add } -k\text{; rearrange terms.}$$

$$3k^2 - k - 4 = 0 \quad \text{Multiply by } -1.$$

$$(3k - 4)(k + 1) = 0 \quad \text{Factor.}$$

$$3k - 4 = 0 \quad \text{or} \quad k + 1 = 0$$

$$k = \frac{4}{3} \qquad k = -1$$

Verify that the solutions are $4/3$ and -1.

TRY YOUR TURN 4

YOUR TURN 4 Solve $\dfrac{1}{x^2 - 4} + \dfrac{2}{x - 2} = \dfrac{1}{x}$.

CAUTION It is possible to get, as a solution of a rational equation, a number that makes one or more of the denominators in the original equation equal to zero. That number is not a solution, so it is *necessary* to check all potential solutions of rational equations. These introduced solutions are called **extraneous solutions**.

EXAMPLE 7 Solving a Rational Equation

Solve $\dfrac{2}{x-3} + \dfrac{1}{x} = \dfrac{6}{x(x-3)}$.

SOLUTION The common denominator is $x(x-3)$. Multiply both sides by $x(x-3)$ and solve the resulting equation.

$$x(x-3)\cdot\left(\frac{2}{x-3}+\frac{1}{x}\right) = x(x-3)\cdot\left[\frac{6}{x(x-3)}\right]$$
$$2x + x - 3 = 6$$
$$3x = 9$$
$$x = 3$$

Checking this potential solution by substitution into the original equation shows that 3 makes two denominators 0. Thus, 3 cannot be a solution, so there is no solution for this equation.

R.4 EXERCISES

Solve each equation.

1. $2x + 8 = x - 4$
2. $5x + 2 = 8 - 3x$
3. $0.2m - 0.5 = 0.1m + 0.7$
4. $\frac{2}{3}k - k + \frac{3}{8} = \frac{1}{2}$
5. $3r + 2 - 5(r + 1) = 6r + 4$
6. $5(a + 3) + 4a - 5 = -(2a - 4)$
7. $2[3m - 2(3 - m) - 4] = 6m - 4$
8. $4[2p - (3 - p) + 5] = -7p - 2$

Solve each equation by factoring or by using the quadratic formula. If the solutions involve square roots, give both the exact solutions and the approximate solutions to three decimal places.

9. $x^2 + 5x + 6 = 0$
10. $x^2 = 3 + 2x$
11. $m^2 = 14m - 49$
12. $2k^2 - k = 10$
13. $12x^2 - 5x = 2$
14. $m(m - 7) = -10$
15. $4x^2 - 36 = 0$
16. $z(2z + 7) = 4$
17. $12y^2 - 48y = 0$
18. $3x^2 - 5x + 1 = 0$
19. $2m^2 - 4m = 3$
20. $p^2 + p - 1 = 0$
21. $k^2 - 10k = -20$
22. $5x^2 - 8x + 2 = 0$
23. $2r^2 - 7r + 5 = 0$
24. $2x^2 - 7x + 30 = 0$
25. $3k^2 + k = 6$
26. $5m^2 + 5m = 0$

Solve each equation.

27. $\dfrac{3x-2}{7} = \dfrac{x+2}{5}$
28. $\dfrac{x}{3} - 7 = 6 - \dfrac{3x}{4}$
29. $\dfrac{4}{x-3} - \dfrac{8}{2x+5} + \dfrac{3}{x-3} = 0$
30. $\dfrac{5}{p-2} - \dfrac{7}{p+2} = \dfrac{12}{p^2-4}$
31. $\dfrac{2m}{m-2} - \dfrac{6}{m} = \dfrac{12}{m^2-2m}$
32. $\dfrac{2y}{y-1} = \dfrac{5}{y} + \dfrac{10-8y}{y^2-y}$
33. $\dfrac{1}{x-2} - \dfrac{3x}{x-1} = \dfrac{2x+1}{x^2-3x+2}$
34. $\dfrac{5}{a} + \dfrac{-7}{a+1} = \dfrac{a^2-2a+4}{a^2+a}$
35. $\dfrac{5}{b+5} - \dfrac{4}{b^2+2b} = \dfrac{6}{b^2+7b+10}$
36. $\dfrac{2}{x^2-2x-3} + \dfrac{5}{x^2-x-6} = \dfrac{1}{x^2+3x+2}$
37. $\dfrac{4}{2x^2+3x-9} + \dfrac{2}{2x^2-x-3} = \dfrac{3}{x^2+4x+3}$

YOUR TURN ANSWERS

1. $-3/2$
2. $3/2, -5$
3. $4 \pm \sqrt{10}$
4. $-1, -4$

R.5 Inequalities

To write that one number is greater than or less than another number, we use the following symbols.

Inequality Symbols

$<$ means *is less than*	$\le$ means *is less than or equal to*
$>$ means *is greater than*	$\ge$ means *is greater than or equal to*

Linear Inequalities

An equation states that two expressions are equal; an **inequality** states that they are unequal. A **linear inequality** is an inequality that can be simplified to the form $ax < b$. (Properties introduced in this section are given only for $<$, but they are equally valid for $>$, $\le$, or $\ge$.) Linear inequalities are solved with the following properties.

Properties of Inequality

For all real numbers a, b, and c:

1. If $a < b$, then $a + c < b + c$.
2. If $a < b$ and if $c > 0$, then $ac < bc$.
3. If $a < b$ and if $c < 0$, then $ac < bc$.

Pay careful attention to property 3; it says that if both sides of an inequality are multiplied by a negative number, the direction of the inequality symbol must be reversed.

EXAMPLE 1 Solving a Linear Inequality

Solve $4 - 3y \le 7 + 2y$.

SOLUTION Use the properties of inequality.

$$4 - 3y + (-4) \le 7 + 2y + (-4) \qquad \text{Add } -4 \text{ to both sides.}$$

$$-3y \le 3 + 2y$$

Remember that *adding* the same number to both sides never changes the direction of the inequality symbol.

$$-3y + (-2y) \le 3 + 2y + (-2y) \qquad \text{Add } -2y \text{ to both sides.}$$

$$-5y \le 3$$

Multiply both sides by $-1/5$. Since $-1/5$ is negative, change the direction of the inequality symbol.

$$-\frac{1}{5}(-5y) \ge -\frac{1}{5}(3)$$

$$y \ge -\frac{3}{5}$$

TRY YOUR TURN 1

YOUR TURN 1 Solve $3z - 2 > 5z + 7$.

CAUTION It is a common error to forget to reverse the direction of the inequality sign when multiplying or dividing by a negative number. For example, to solve $-4x \le 12$, we must multiply by $-1/4$ on both sides *and* reverse the inequality symbol to get $x \ge -3$.

The solution $y \geq -3/5$ in Example 1 represents an interval on the number line. **Interval notation** often is used for writing intervals. With interval notation, $y \geq -3/5$ is written as $[-3/5, \infty)$. This is an example of a **half-open interval**, since one endpoint, $-3/5$, is included. The **open interval** $(2, 5)$ corresponds to $2 < x < 5$, with neither endpoint included. The **closed interval** $[2, 5]$ includes both endpoints and corresponds to $2 \leq x \leq 5$.

The **graph** of an interval shows all points on a number line that correspond to the numbers in the interval. To graph the interval $[-3/5, \infty)$, for example, use a solid circle at $-3/5$, since $-3/5$ is part of the solution. To show that the solution includes all real numbers greater than or equal to $-3/5$, draw a heavy arrow pointing to the right (the positive direction). See Figure 1.

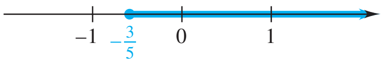


FIGURE 1

EXAMPLE 2 Graphing a Linear Inequality

Solve $-2 < 5 + 3m < 20$. Graph the solution.

SOLUTION The inequality $-2 < 5 + 3m < 20$ says that $5 + 3m$ is *between* -2 and 20. Solve this inequality with an extension of the properties given above. Work as follows, first adding -5 to each part.

$$-2 + (-5) < 5 + 3m + (-5) < 20 + (-5)$$
$$-7 < 3m < 15$$

Now multiply each part by 1/3.

$$-\frac{7}{3} < m < 5$$

A graph of the solution is given in Figure 2; here open circles are used to show that $-7/3$ and 5 are *not* part of the graph.*

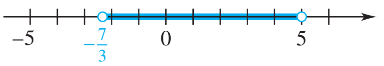


FIGURE 2

Quadratic Inequalities

A **quadratic inequality** has the form $ax^2 + bx + c > 0$ (or $<$, or $\leq$, or $\geq$). The greatest exponent is 2. The next few examples show how to solve quadratic inequalities.

EXAMPLE 3 Solving a Quadratic Inequality

Solve the quadratic inequality $x^2 - x < 12$.

SOLUTION Write the inequality with 0 on one side, as $x^2 - x - 12 < 0$. This inequality is solved with values of x that make $x^2 - x - 12$ negative (< 0). The quantity $x^2 - x - 12$ changes from positive to negative or from negative to positive at the points where it equals 0. For this reason, first solve the *equation* $x^2 - x - 12 = 0$.

$$x^2 - x - 12 = 0$$
$$(x - 4)(x + 3) = 0$$
$$x = 4 \quad \text{or} \quad x = -3$$

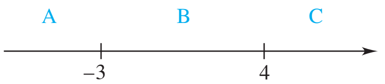


FIGURE 3

Locating -3 and 4 on a number line, as shown in Figure 3, determines three intervals A, B, and C. Decide which intervals include numbers that make $x^2 - x - 12$ negative by substituting any number from each interval into the polynomial. For example,

$$\text{choose } -4 \text{ from interval A: } (-4)^2 - (-4) - 12 = 8 > 0;$$
$$\text{choose 0 from interval B: } 0^2 - 0 - 12 = -12 < 0;$$
$$\text{choose 5 from interval C: } 5^2 - 5 - 12 = 8 > 0.$$

Only numbers in interval B satisfy the given inequality, so the solution is $(-3, 4)$. A graph of this solution is shown in Figure 4.

TRY YOUR TURN 2

FIGURE 4

YOUR TURN 2 Solve $3y^2 \leq 16y + 12$.

*Some textbooks use brackets in place of solid circles for the graph of a closed interval, and parentheses in place of open circles for the graph of an open interval.

EXAMPLE 4 Solving a Polynomial Inequality

Solve the inequality $x^3 + 2x^2 - 3x \geq 0$.

SOLUTION This is not a quadratic inequality because of the x^3 term, but we solve it in a similar way by first factoring the polynomial.

$$x^3 + 2x^2 - 3x = x(x^2 + 2x - 3) \quad \text{Factor out the common factor.}$$
$$= x(x - 1)(x + 3) \quad \text{Factor the quadratic.}$$

Now solve the corresponding equation.

$$x(x - 1)(x + 3) = 0$$
$$x = 0 \quad \text{or} \quad x - 1 = 0 \quad \text{or} \quad x + 3 = 0$$
$$x = 1 \qquad\qquad x = -3$$

FIGURE 5

These three solutions determine four intervals on the number line: $(-\infty, -3)$, $(-3, 0)$, $(0, 1)$, and $(1, \infty)$. Substitute a number from each interval into the original inequality to determine that the solution consists of the numbers between -3 and 0 (including the endpoints) and all numbers that are greater than or equal to 1. See Figure 5. In interval notation, the solution is

$$[-3, 0] \cup [1, \infty).*$$

Inequalities with Fractions

Inequalities with fractions are solved in a similar manner as quadratic inequalities.

EXAMPLE 5 Solving a Rational Inequality

Solve $\dfrac{2x - 3}{x} \geq 1$.

SOLUTION First solve the corresponding equation.

$$\frac{2x - 3}{x} = 1$$
$$2x - 3 = x \quad \text{Multiply both sides by } x.$$
$$x = 3 \quad \text{Solve for } x.$$

The solution, $x = 3$, determines the intervals on the number line where the fraction may change from greater than 1 to less than 1. This change also may occur on either side of a number that makes the denominator equal 0. Here, the x-value that makes the denominator 0 is $x = 0$. Test each of the three intervals determined by the numbers 0 and 3.

$$\text{For } (-\infty, 0), \text{ choose } -1\text{: } \frac{2(-1) - 3}{-1} = 5 \geq 1.$$

$$\text{For } (0, 3), \text{ choose } 1\text{: } \frac{2(1) - 3}{1} = -1 \not\geq 1.$$

$$\text{For } (3, \infty), \text{ choose } 4\text{: } \frac{2(4) - 3}{4} = \frac{5}{4} \geq 1.$$

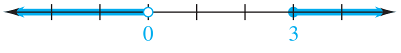


FIGURE 6

The symbol $\not\geq$ means "is *not* greater than or equal to." Testing the endpoints 0 and 3 shows that the solution is $(-\infty, 0) \cup [3, \infty)$, as shown in Figure 6.

CAUTION A common error is to try to solve the inequality in Example 5 by multiplying both sides by x. The reason this is wrong is that we don't know in the beginning whether x is positive, negative, or 0. If x is negative, the $\geq$ would change to $\leq$ according to the third property of inequality listed at the beginning of this section.

*The symbol $\cup$ indicates the *union* of two sets, which includes all elements in either set.

EXAMPLE 6 Solving a Rational Inequality

Solve $\dfrac{(x-1)(x+1)}{x} \le 0.$

SOLUTION We first solve the corresponding equation.

$$\frac{(x-1)(x+1)}{x} = 0$$

$$(x-1)(x+1) = 0 \qquad \text{Multiply both sides by } x.$$

$$x = 1 \quad \text{or} \quad x = -1 \qquad \text{Use the zero-factor property.}$$

Setting the denominator equal to 0 gives $x = 0$, so the intervals of interest are $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$. Testing a number from each region in the original inequality and checking the endpoints, we find the solution is

$$(-\infty, -1] \cup (0, 1],$$

as shown in Figure 7.

FIGURE 7

CAUTION Remember to solve the equation formed by setting the *denominator* equal to zero. Any number that makes the denominator zero always creates two intervals on the number line. For instance, in Example 6, substituting $x = 0$ makes the denominator of the rational inequality equal to 0, so we know that there may be a sign change from one side of 0 to the other (as was indeed the case).

EXAMPLE 7 Solving a Rational Inequality

Solve $\dfrac{x^2 - 3x}{x^2 - 9} < 4.$

SOLUTION Solve the corresponding equation.

$$\frac{x^2 - 3x}{x^2 - 9} = 4$$

$$x^2 - 3x = 4x^2 - 36 \qquad \text{Multiply by } x^2 - 9.$$

$$0 = 3x^2 + 3x - 36 \qquad \text{Get 0 on one side.}$$

$$0 = x^2 + x - 12 \qquad \text{Multiply by } \tfrac{1}{3}.$$

$$0 = (x+4)(x-3) \qquad \text{Factor.}$$

$$x = -4 \quad \text{or} \quad x = 3$$

Now set the denominator equal to 0 and solve that equation.

$$x^2 - 9 = 0$$

$$(x-3)(x+3) = 0$$

$$x = 3 \quad \text{or} \quad x = -3$$

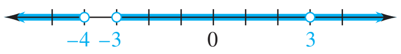


FIGURE 8

The intervals determined by the three (different) solutions are $(-\infty, -4)$, $(-4, -3)$, $(-3, 3)$, and $(3, \infty)$. Testing a number from each interval in the given inequality shows that the solution is

$$(-\infty, -4) \cup (-3, 3) \cup (3, \infty),$$

as shown in Figure 8. For this example, none of the endpoints are part of the solution because $x = 3$ and $x = -3$ make the denominator zero and $x = -4$ produces an equality.

TRY YOUR TURN 3

YOUR TURN 3 Solve $\dfrac{k^2 - 35}{k} \ge 2.$

R.5 EXERCISES

Write each expression in interval notation. Graph each interval.

1. $x < 4$

2. $x \geq -3$

3. $1 \leq x < 2$

4. $-2 \leq x \leq 3$

5. $-9 > x$

6. $6 \leq x$

Using the variable *x*, write each interval as an inequality.

7. $[-7, -3]$

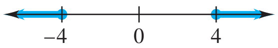
8. $[4, 10)$

9. $(-\infty, -1]$

10. $(3, \infty)$

11.


12.


13.


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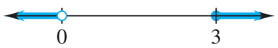

Solve each inequality and graph the solution.

15. $6p + 7 \leq 19$

16. $6k - 4 < 3k - 1$

17. $m - (3m - 2) + 6 < 7m - 19$

18. $-2(3y - 8) \geq 5(4y - 2)$

19. $3p - 1 < 6p + 2(p - 1)$

20. $x + 5(x + 1) > 4(2 - x) + x$

21. $-11 < y - 7 < -1$

22. $8 \leq 3r + 1 \leq 13$

23. $-2 < \frac{1 - 3k}{4} \leq 4$

24. $-1 \leq \frac{5y + 2}{3} \leq 4$

25. $\frac{3}{5}(2p + 3) \geq \frac{1}{10}(5p + 1)$

26. $\frac{8}{3}(z - 4) \leq \frac{2}{9}(3z + 2)$

Solve each inequality. Graph each solution.

27. $(m - 3)(m + 5) < 0$

28. $(t + 6)(t - 1) \geq 0$

29. $y^2 - 3y + 2 < 0$

30. $2k^2 + 7k - 4 > 0$

31. $x^2 - 16 > 0$

32. $2k^2 - 7k - 15 \leq 0$

33. $x^2 - 4x \geq 5$

34. $10r^2 + r \leq 2$

35. $3x^2 + 2x > 1$

36. $3a^2 + a > 10$

37. $9 - x^2 \leq 0$

38. $p^2 - 16p > 0$

39. $x^3 - 4x \geq 0$

40. $x^3 + 7x^2 + 12x \leq 0$

41. $2x^3 - 14x^2 + 12x < 0$

42. $3x^3 - 9x^2 - 12x > 0$

Solve each inequality.

43. $\frac{m - 3}{m + 5} \leq 0$

44. $\frac{r + 1}{r - 1} > 0$

45. $\frac{k - 1}{k + 2} > 1$

46. $\frac{a - 5}{a + 2} < -1$

47. $\frac{2y + 3}{y - 5} \leq 1$

48. $\frac{a + 2}{3 + 2a} \leq 5$

49. $\frac{2k}{k - 3} \leq \frac{4}{k - 3}$

50. $\frac{5}{p + 1} > \frac{12}{p + 1}$

51. $\frac{2x}{x^2 - x - 6} \geq 0$

52. $\frac{8}{p^2 + 2p} > 1$

53. $\frac{z^2 + z}{z^2 - 1} \geq 3$

54. $\frac{a^2 + 2a}{a^2 - 4} \leq 2$

YOUR TURN ANSWERS

1. $z < -9/2$ **2.** $[-2/3, 6]$ **3.** $[-5, 0) \cup [7, \infty)$

R.6 Exponents

Integer Exponents

Recall that $a^2 = a \cdot a$, while $a^3 = a \cdot a \cdot a$, and so on. In this section, a more general meaning is given to the symbol a^n.

Definition of Exponent

If n is a natural number, then

$$a^n = a \cdot a \cdot a \cdot \cdots \cdot a,$$

where a appears as a factor n times.

In the expression a^n, the power n is the **exponent** and a is the **base**. This definition can be extended by defining a^n for zero and negative integer values of n.

Zero and Negative Exponents

If a is any nonzero real number, and if n is a positive integer, then

$$a^0 = 1 \quad \text{and} \quad a^{-n} = \frac{1}{a^n}.$$

(The symbol 0^0 is meaningless.)

EXAMPLE 1 Exponents

YOUR TURN 1 Find $\left(\frac{2}{3}\right)^{-3}$.

(a) $6^0 = 1$

(b) $(-9)^0 = 1$

(c) $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

(d) $9^{-1} = \frac{1}{9^1} = \frac{1}{9}$

(e) $\left(\frac{3}{4}\right)^{-1} = \frac{1}{(3/4)^1} = \frac{1}{3/4} = \frac{4}{3}$

TRY YOUR TURN 1

The following properties follow from the definitions of exponents given above.

Properties of Exponents

For any integers m and n, and any real numbers a and b for which the following exist:

1. $a^m \cdot a^n = a^{m+n}$
2. $\frac{a^m}{a^n} = a^{m-n}$
3. $(a^m)^n = a^{mn}$
4. $(ab)^m = a^m \cdot b^m$
5. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Note that $(-a)^n = a^n$ if n is an even integer, but $(-a)^n = -a^n$ if n is an odd integer.

EXAMPLE 2 Simplifying Exponential Expressions

Use the properties of exponents to simplify each expression. Leave answers with positive exponents. Assume that all variables represent positive real numbers.

(a) $7^4 \cdot 7^6 = 7^{4+6} = 7^{10}$ (or 282,475,249) Property 1

(b) $\frac{9^{14}}{9^6} = 9^{14-6} = 9^8$ (or 43,046,721) Property 2

(c) $\frac{r^9}{r^{17}} = r^{9-17} = r^{-8} = \frac{1}{r^8}$ Property 2

(d) $(2m^3)^4 = 2^4 \cdot (m^3)^4 = 16m^{12}$ Properties 3 and 4

(e) $(3x)^4 = 3^4 \cdot x^4 = 81x^4$ Property 4

(f) $\left(\frac{x^2}{y^3}\right)^6 = \frac{(x^2)^6}{(y^3)^6} = \frac{x^{2\cdot 6}}{y^{3\cdot 6}} = \frac{x^{12}}{y^{18}}$ Properties 3 and 5

(g) $\frac{a^{-3}b^5}{a^4b^{-7}} = \frac{b^{5-(-7)}}{a^{4-(-3)}} = \frac{b^{5+7}}{a^{4+3}} = \frac{b^{12}}{a^7}$ Property 2

(h) $p^{-1} + q^{-1} = \frac{1}{p} + \frac{1}{q}$ — Definition of a^{-n}.

$= \frac{1}{p} \cdot \frac{q}{q} + \frac{1}{q} \cdot \frac{p}{p}$ — Get common denominator.

$= \frac{q}{pq} + \frac{p}{pq} = \frac{p + q}{pq}$ — Add.

(i) $\frac{x^{-2} - y^{-2}}{x^{-1} - y^{-1}} = \frac{\frac{1}{x^2} - \frac{1}{y^2}}{\frac{1}{x} - \frac{1}{y}}$ — Definition of a^{-n}

$= \frac{\frac{y^2 - x^2}{x^2y^2}}{\frac{y - x}{xy}}$ — Get common denominators and combine terms.

$= \frac{y^2 - x^2}{x^2y^2} \cdot \frac{xy}{y - x}$ — Invert and multiply.

$= \frac{(y - x)(y + x)}{x^2y^2} \cdot \frac{xy}{y - x}$ — Factor.

$= \frac{x + y}{xy}$ — Simplify.

TRY YOUR TURN 2

YOUR TURN 2 Simplify $\left(\frac{y^2z^{-4}}{y^{-3}z^4}\right)^{-2}$.

CAUTION If Example 2(e) were written $3x^4$, the properties of exponents would not apply. When no parentheses are used, the exponent refers only to the factor closest to it. Also notice in Examples 2(c), 2(g), 2(h), and 2(i) that a negative exponent does *not* indicate a negative number.

Roots

For *even* values of n and nonnegative values of a, the expression $a^{1/n}$ is defined to be the **positive** n**th root** of a or the **principal** n**th root** of a. For example, $a^{1/2}$ denotes the positive second root, or **square root**, of a, while $a^{1/4}$ is the positive fourth root of a. When n is *odd*, there is only one nth root, which has the same sign as a. For example, $a^{1/3}$, the **cube root** of a, has the same sign as a. By definition, if $b = a^{1/n}$, then $b^n = a$. On a calculator, a number is raised to a power using a key labeled x^y, y^x, or $\wedge$. For example, to take the fourth root of 6 on a TI-84 Plus C calculator, enter $6 \wedge (1/4)$ to get the result 1.56508458.

EXAMPLE 3 Calculations with Exponents

(a) $121^{1/2} = 11$, since 11 is positive and $11^2 = 121$.

(b) $625^{1/4} = 5$, since $5^4 = 625$.

(c) $256^{1/4} = 4$

(d) $64^{1/6} = 2$

(e) $27^{1/3} = 3$

(f) $(-32)^{1/5} = -2$

(g) $128^{1/7} = 2$

(h) $(-49)^{1/2}$ is not a real number.

TRY YOUR TURN 3

YOUR TURN 3 Find $125^{1/3}$.

Rational Exponents

In the following definition, the domain of an exponent is extended to include all rational numbers.

Definition of $a^{m/n}$

For all real numbers a for which the indicated roots exist, and for any rational number m/n,

$$a^{m/n} = (a^{1/n})^m.$$

EXAMPLE 4 Calculations with Exponents

YOUR TURN 4 Find $16^{-3/4}$.

(a) $27^{2/3} = (27^{1/3})^2 = 3^2 = 9$

(b) $32^{2/5} = (32^{1/5})^2 = 2^2 = 4$

(c) $64^{4/3} = (64^{1/3})^4 = 4^4 = 256$

(d) $25^{3/2} = (25^{1/2})^3 = 5^3 = 125$

TRY YOUR TURN 4

NOTE $27^{2/3}$ could also be evaluated as $(27^2)^{1/3}$, but this is more difficult to perform without a calculator because it involves squaring 27 and then taking the cube root of this large number. On the other hand, when we evaluate it as $(27^{1/3})^2$, we know that the cube root of 27 is 3 without using a calculator, and squaring 3 is easy.

All the properties for integer exponents given in this section also apply to any rational exponent on a nonnegative real-number base.

EXAMPLE 5 Simplifying Exponential Expressions

YOUR TURN 5 Simplify $\left(\frac{x^{1/2}x^4}{x^{3/2}}\right)^{1/3}$.

(a) $\frac{y^{1/3}y^{5/3}}{y^3} = \frac{y^{1/3+5/3}}{y^3} = \frac{y^2}{y^3} = y^{2-3} = y^{-1} = \frac{1}{y}$

(b) $m^{2/3}(m^{7/3} + 2m^{1/3}) = m^{2/3+7/3} + 2m^{2/3+1/3} = m^3 + 2m$

(c) $\left(\frac{m^7n^{-2}}{m^{-5}n^2}\right)^{1/4} = \left(\frac{m^{7-(-5)}}{n^{2-(-2)}}\right)^{1/4} = \left(\frac{m^{12}}{n^4}\right)^{1/4} = \frac{(m^{12})^{1/4}}{(n^4)^{1/4}} = \frac{m^{12/4}}{n^{4/4}} = \frac{m^3}{n}$

TRY YOUR TURN 5

In calculus, it is often necessary to factor expressions involving fractional exponents.

EXAMPLE 6 Simplifying Exponential Expressions

Factor out the smallest power of the variable, assuming all variables represent positive real numbers.

(a) $4m^{1/2} + 3m^{3/2}$

SOLUTION The smallest exponent is 1/2. Factoring out $m^{1/2}$ yields

$$\begin{aligned} 4m^{1/2} + 3m^{3/2} &= m^{1/2}(4m^{1/2-1/2} + 3m^{3/2-1/2}) \\ &= m^{1/2}(4 + 3m). \end{aligned}$$

Check this result by multiplying $m^{1/2}$ by $4 + 3m$.

(b) $9x^{-2} - 6x^{-3}$

SOLUTION The smallest exponent here is -3. Since 3 is a common numerical factor, factor out $3x^{-3}$.

$$9x^{-2} - 6x^{-3} = 3x^{-3}(3x^{-2-(-3)} - 2x^{-3-(-3)}) = 3x^{-3}(3x - 2)$$

Check by multiplying. The factored form can be written without negative exponents as

$$\frac{3(3x - 2)}{x^3}.$$

(c) $(x^2+5)(3x-1)^{-1/2}(2)+(3x-1)^{1/2}(2x)$

SOLUTION There is a common factor of 2. Also, $(3x-1)^{-1/2}$ and $(3x-1)^{1/2}$ have a common factor. Always factor out the quantity to the *smallest* exponent. Here $-1/2 < 1/2$, so the common factor is $2(3x-1)^{-1/2}$ and the factored form is

$$2(3x-1)^{-1/2}[(x^2+5)+(3x-1)x] = 2(3x-1)^{-1/2}(4x^2-x+5).$$

TRY YOUR TURN 6

YOUR TURN 6 Factor $5z^{1/3}+4z^{-2/3}$.

R.6 EXERCISES

Evaluate each expression. Write all answers without exponents.

1. 8^{-2}

2. 3^{-4}

3. 5^0

4. $\left(-\frac{3}{4}\right)^0$

5. $-(-3)^{-2}$

6. $-(-3^{-2})$

7. $\left(\frac{1}{6}\right)^{-2}$

8. $\left(\frac{4}{3}\right)^{-3}$

Simplify each expression. Assume that all variables represent positive real numbers. Write answers with only positive exponents.

9. $\frac{4^{-2}}{4}$

10. $\frac{8^9 \cdot 8^{-7}}{8^{-3}}$

11. $\frac{10^8 \cdot 10^{-10}}{10^4 \cdot 10^2}$

12. $\left(\frac{7^{-12} \cdot 7^3}{7^{-8}}\right)^{-1}$

13. $\frac{x^4 \cdot x^3}{x^5}$

14. $\frac{y^{10} \cdot y^{-4}}{y^6}$

15. $\frac{(4k^{-1})^2}{2k^{-5}}$

16. $\frac{(3z^2)^{-1}}{z^5}$

17. $\frac{3^{-1} \cdot x \cdot y^2}{x^{-4} \cdot y^5}$

18. $\frac{5^{-2}m^2y^{-2}}{5^2m^{-1}y^{-2}}$

19. $\left(\frac{a^{-1}}{b^2}\right)^{-3}$

20. $\left(\frac{c^3}{7d^{-2}}\right)^{-2}$

Simplify each expression, writing the answer as a single term without negative exponents.

21. $a^{-1}+b^{-1}$

22. $b^{-2}-a$

23. $\frac{2n^{-1}-2m^{-1}}{m+n^2}$

24. $\left(\frac{m}{3}\right)^{-1}+\left(\frac{n}{2}\right)^{-2}$

25. $(x^{-1}-y^{-1})^{-1}$

26. $(x \cdot y^{-1}-y^{-2})^{-2}$

Write each number without exponents.

27. $121^{1/2}$

28. $27^{1/3}$

29. $32^{2/5}$

30. $-125^{2/3}$

31. $\left(\frac{36}{144}\right)^{1/2}$

32. $\left(\frac{64}{27}\right)^{1/3}$

33. $8^{-4/3}$

34. $625^{-1/4}$

35. $\left(\frac{27}{64}\right)^{-1/3}$

36. $\left(\frac{121}{100}\right)^{-3/2}$

Simplify each expression. Write all answers with only positive exponents. Assume that all variables represent positive real numbers.

37. $3^{2/3} \cdot 3^{4/3}$

38. $27^{2/3} \cdot 27^{-1/3}$

39. $\frac{4^{9/4} \cdot 4^{-7/4}}{4^{-10/4}}$

40. $\frac{3^{-5/2} \cdot 3^{3/2}}{3^{7/2} \cdot 3^{-9/2}}$

41. $\left(\frac{x^6y^{-3}}{x^{-2}y^5}\right)^{1/2}$

42. $\left(\frac{a^{-7}b^{-1}}{b^{-4}a^2}\right)^{1/3}$

43. $\frac{7^{-1/3} \cdot 7r^{-3}}{7^{2/3} \cdot (r^{-2})^2}$

44. $\frac{12^{3/4} \cdot 12^{5/4} \cdot y^{-2}}{12^{-1} \cdot (y^{-3})^{-2}}$

45. $\frac{3k^2 \cdot (4k^{-3})^{-1}}{4^{1/2} \cdot k^{7/2}}$

46. $\frac{8p^{-3} \cdot (4p^2)^{-2}}{p^{-5}}$

47. $\frac{a^{4/3} \cdot b^{1/2}}{a^{2/3} \cdot b^{-3/2}}$

48. $\frac{x^{3/2} \cdot y^{4/5} \cdot z^{-3/4}}{x^{5/3} \cdot y^{-6/5} \cdot z^{1/2}}$

49. $\frac{k^{-3/5} \cdot h^{-1/3} \cdot t^{2/5}}{k^{-1/5} \cdot h^{-2/3} \cdot t^{1/5}}$

50. $\frac{m^{7/3} \cdot n^{-2/5} \cdot p^{3/8}}{m^{-2/3} \cdot n^{3/5} \cdot p^{-5/8}}$

Factor each expression.

51. $3x^3(x^2+3x)^2-15x(x^2+3x)^2$

52. $6x(x^3+7)^2-6x^2(3x^2+5)(x^3+7)$

53. $10x^3(x^2-1)^{-1/2}-5x(x^2-1)^{1/2}$

54. $9(6x+2)^{1/2}+3(9x-1)(6x+2)^{-1/2}$

55. $x(2x+5)^2(x^2-4)^{-1/2}+2(x^2-4)^{1/2}(2x+5)$

56. $(4x^2+1)^2(2x-1)^{-1/2}+16x(4x^2+1)(2x-1)^{1/2}$

YOUR TURN ANSWERS

1. $27/8$ **2.** z^{16}/y^{10} **3.** 5

4. $1/8$ **5.** x **6.** $z^{-2/3}(5z+4)$

R.7 Radicals

We have defined $a^{1/n}$ as the positive or principal nth root of a for appropriate values of a and n. An alternative notation for $a^{1/n}$ uses radicals.

Radicals

If n is an even natural number and $a > 0$, or n is an odd natural number, then

$$a^{1/n} = \sqrt[n]{a}.$$

The symbol $\sqrt[n]{\ }$ is a **radical sign**, the number a is the **radicand**, and n is the **index** of the radical. The familiar symbol $\sqrt{a}$ is used instead of $\sqrt[2]{a}$.

EXAMPLE 1 Radical Calculations

(a) $\sqrt[4]{16} = 16^{1/4} = 2$

(b) $\sqrt[5]{-32} = -2$

(c) $\sqrt[3]{1000} = 10$

(d) $\sqrt[6]{\dfrac{64}{729}} = \dfrac{2}{3}$

With $a^{1/n}$ written as $\sqrt[n]{a}$, the expression $a^{m/n}$ also can be written using radicals.

$$a^{m/n} = (\sqrt[n]{a})^m \quad \text{or} \quad a^{m/n} = \sqrt[n]{a^m}$$

The following properties of radicals depend on the definitions and properties of exponents.

Properties of Radicals

For all real numbers a and b and natural numbers m and n such that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers:

1. $(\sqrt[n]{a})^n = a$
2. $\sqrt[n]{a^n} = \begin{cases} |a| & \text{if } n \text{ is even} \\ a & \text{if } n \text{ is odd} \end{cases}$
3. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$
4. $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}} \quad (b \neq 0)$
5. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

Property 3 can be used to simplify certain radicals. For example, since $48 = 16 \cdot 3$,

$$\sqrt{48} = \sqrt{16 \cdot 3} = \sqrt{16} \cdot \sqrt{3} = 4\sqrt{3}.$$

To some extent, simplification is in the eye of the beholder, and $\sqrt{48}$ might be considered as simple as $4\sqrt{3}$. In this textbook, we will consider an expression to be simpler when we have removed as many factors as possible from under the radical.

EXAMPLE 2 Radical Calculations

(a) $\sqrt{1000} = \sqrt{100 \cdot 10} = \sqrt{100} \cdot \sqrt{10} = 10\sqrt{10}$

(b) $\sqrt{128} = \sqrt{64 \cdot 2} = 8\sqrt{2}$

(c) $\sqrt{2} \cdot \sqrt{18} = \sqrt{2 \cdot 18} = \sqrt{36} = 6$

(d) $\sqrt[3]{54} = \sqrt[3]{27 \cdot 2} = \sqrt[3]{27} \cdot \sqrt[3]{2} = 3\sqrt[3]{2}$

(e) $\sqrt{288m^5} = \sqrt{144 \cdot m^4 \cdot 2m} = 12m^2\sqrt{2m}$

(f) $2\sqrt{18} - 5\sqrt{32} = 2\sqrt{9 \cdot 2} - 5\sqrt{16 \cdot 2}$

$= 2\sqrt{9} \cdot \sqrt{2} - 5\sqrt{16} \cdot \sqrt{2}$

$= 2(3)\sqrt{2} - 5(4)\sqrt{2} = -14\sqrt{2}$

(g) $\sqrt{x^5} \cdot \sqrt[3]{x^5} = x^{5/2} \cdot x^{5/3} = x^{5/2+5/3} = x^{25/6} = \sqrt[6]{x^{25}} = x^4\sqrt[6]{x}$

YOUR TURN 1 Simplify $\sqrt{28x^9y^5}$.

TRY YOUR TURN 1

When simplifying a square root, keep in mind that $\sqrt{x}$ is nonnegative by definition. Also, $\sqrt{x^2}$ is not x, but $|x|$, the **absolute value of x**, defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

For example, $\sqrt{(-5)^2} = |-5| = 5$. It is correct, however, to simplify $\sqrt{x^4} = x^2$. We need not write $|x^2|$ because x^2 is always nonnegative.

EXAMPLE 3 Simplifying by Factoring

Simplify $\sqrt{m^2 - 4m + 4}$.

SOLUTION Factor the polynomial as $m^2 - 4m + 4 = (m - 2)^2$. Then by property 2 of radicals and the definition of absolute value,

$$\sqrt{(m-2)^2} = |m - 2| = \begin{cases} m - 2 & \text{if } m - 2 \geq 0 \\ -(m - 2) = 2 - m & \text{if } m - 2 < 0. \end{cases}$$

CAUTION Avoid the common error of writing $\sqrt{a^2 + b^2}$ as $\sqrt{a^2} + \sqrt{b^2}$. We must add a^2 and b^2 *before* taking the square root. For example, $\sqrt{16 + 9} = \sqrt{25} = 5$, *not* $\sqrt{16} + \sqrt{9} = 4 + 3 = 7$. This idea applies as well to higher roots. For example, in general,

$$\sqrt[3]{a^3 + b^3} \neq \sqrt[3]{a^3} + \sqrt[3]{b^3},$$

$$\sqrt[4]{a^4 + b^4} \neq \sqrt[4]{a^4} + \sqrt[4]{b^4}.$$

Also, $$\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}.$$

Rationalizing Denominators

The next example shows how to *rationalize* (remove all radicals from) the denominator in an expression containing radicals.

EXAMPLE 4 Rationalizing Denominators

Simplify each expression by rationalizing the denominator.

(a) $\dfrac{4}{\sqrt{3}}$

SOLUTION To rationalize the denominator, multiply by $\sqrt{3}/\sqrt{3}$ (or 1) so the denominator of the product is a rational number.

$$\frac{4}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{4\sqrt{3}}{3} \qquad \sqrt{3} \cdot \sqrt{3} = \sqrt{9} = 3$$

(b) $\dfrac{2}{\sqrt[3]{x}}$

SOLUTION Here, we need a perfect cube under the radical sign to rationalize the denominator. Multiplying by $\sqrt[3]{x^2}/\sqrt[3]{x^2}$ gives

$$\frac{2}{\sqrt[3]{x}} \cdot \frac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}} = \frac{2\sqrt[3]{x^2}}{\sqrt[3]{x^3}} = \frac{2\sqrt[3]{x^2}}{x}.$$

(c) $\dfrac{1}{1 - \sqrt{2}}$

SOLUTION The best approach here is to multiply both numerator and denominator by the number $1 + \sqrt{2}$. The expressions $1 + \sqrt{2}$ and $1 - \sqrt{2}$ are conjugates,* and their product is $1^2 - (\sqrt{2})^2 = 1 - 2 = -1$. Thus,

$$\frac{1}{1 - \sqrt{2}} = \frac{1(1 + \sqrt{2})}{(1 - \sqrt{2})(1 + \sqrt{2})} = \frac{1 + \sqrt{2}}{1 - 2} = -1 - \sqrt{2}.$$

TRY YOUR TURN 2

YOUR TURN 2 Rationalize the denominator in $\dfrac{5}{\sqrt{x} - \sqrt{y}}$.

Sometimes it is advantageous to rationalize the *numerator* of a rational expression. The following example arises in calculus when evaluating a *limit.*

EXAMPLE 5 Rationalizing Numerators

Rationalize each numerator.

(a) $\dfrac{\sqrt{x} - 3}{x - 9}$.

SOLUTION Multiply the numerator and denominator by the conjugate of the numerator, $\sqrt{x} + 3$.

$$\frac{\sqrt{x} - 3}{x - 9} \cdot \frac{\sqrt{x} + 3}{\sqrt{x} + 3} = \frac{(\sqrt{x})^2 - 3^2}{(x - 9)(\sqrt{x} + 3)} \qquad (a - b)(a + b) = a^2 - b^2$$

$$= \frac{x - 9}{(x - 9)(\sqrt{x} + 3)}$$

$$= \frac{1}{\sqrt{x} + 3}$$

*If a and b are real numbers, the *conjugate* of $a + b$ is $a - b$.

(b) $\dfrac{\sqrt{3}+\sqrt{x+3}}{\sqrt{3}-\sqrt{x+3}}$

SOLUTION Multiply the numerator and denominator by the conjugate of the numerator, $\sqrt{3}-\sqrt{x+3}$.

$$\frac{\sqrt{3}+\sqrt{x+3}}{\sqrt{3}-\sqrt{x+3}}\cdot\frac{\sqrt{3}-\sqrt{x+3}}{\sqrt{3}-\sqrt{x+3}}=\frac{3-(x+3)}{3-2\sqrt{3}\sqrt{x+3}+(x+3)}$$

$$=\frac{-x}{6+x-2\sqrt{3(x+3)}}$$

TRY YOUR TURN 3

YOUR TURN 3 Rationalize the numerator in $\dfrac{4+\sqrt{x}}{16-x}$.

R.7 EXERCISES

Simplify each expression by removing as many factors as possible from under the radical. Assume that all variables represent positive real numbers.

1. $\sqrt[3]{125}$
2. $\sqrt[4]{1296}$
3. $\sqrt[5]{-3125}$
4. $\sqrt{50}$
5. $\sqrt{2000}$
6. $\sqrt{32y^5}$
7. $\sqrt{27}\cdot\sqrt{3}$
8. $\sqrt{2}\cdot\sqrt{32}$
9. $7\sqrt{2}-8\sqrt{18}+4\sqrt{72}$
10. $4\sqrt{3}-5\sqrt{12}+3\sqrt{75}$
11. $4\sqrt{7}-\sqrt{28}+\sqrt{343}$
12. $3\sqrt{28}-4\sqrt{63}+\sqrt{112}$
13. $\sqrt[3]{2}-\sqrt[3]{16}+2\sqrt[3]{54}$
14. $2\sqrt[3]{5}-4\sqrt[3]{40}+3\sqrt[3]{135}$
15. $\sqrt{2x^3y^2z^4}$
16. $\sqrt{160r^7s^9t^{12}}$
17. $\sqrt[3]{128x^3y^8z^9}$
18. $\sqrt[4]{x^8y^7z^{11}}$
19. $\sqrt{a^3b^5}-2\sqrt{a^7b^3}+\sqrt{a^3b^9}$
20. $\sqrt{p^7q^3}-\sqrt{p^5q^9}+\sqrt{p^9q}$
21. $\sqrt{a}\cdot\sqrt[3]{a}$
22. $\sqrt{b^3}\cdot\sqrt[4]{b^3}$

Simplify each root, if possible.

23. $\sqrt{16-8x+x^2}$
24. $\sqrt{9y^2+30y+25}$
25. $\sqrt{4-25z^2}$
26. $\sqrt{9k^2+h^2}$

Rationalize each denominator. Assume that all radicands represent positive real numbers.

27. $\dfrac{5}{\sqrt{7}}$
28. $\dfrac{5}{\sqrt{10}}$
29. $\dfrac{-3}{\sqrt{12}}$
30. $\dfrac{4}{\sqrt{8}}$
31. $\dfrac{3}{1-\sqrt{2}}$
32. $\dfrac{5}{2-\sqrt{6}}$
33. $\dfrac{6}{2+\sqrt{2}}$
34. $\dfrac{\sqrt{5}}{\sqrt{5}+\sqrt{2}}$
35. $\dfrac{1}{\sqrt{r}-\sqrt{3}}$
36. $\dfrac{5}{\sqrt{m}-\sqrt{5}}$
37. $\dfrac{y-5}{\sqrt{y}-\sqrt{5}}$
38. $\dfrac{\sqrt{z}-1}{\sqrt{z}-\sqrt{5}}$
39. $\dfrac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}}$
40. $\dfrac{\sqrt{p}+\sqrt{p^2-1}}{\sqrt{p}-\sqrt{p^2-1}}$

Rationalize each numerator. Assume that all radicands represent positive real numbers.

41. $\dfrac{1+\sqrt{2}}{2}$
42. $\dfrac{3-\sqrt{3}}{6}$
43. $\dfrac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}}$
44. $\dfrac{\sqrt{p}-\sqrt{p-2}}{\sqrt{p}}$

YOUR TURN ANSWERS

1. $2x^4y^2\sqrt{7xy}$ **2.** $5(\sqrt{x}+\sqrt{y})/(x-y)$ **3.** $1/(4-\sqrt{x})$

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1 Linear Functions

Over short time intervals, many changes in the economy are well modeled by linear functions. In an exercise in the first section of this chapter, we will examine a linear model that predicts the number of cellular telephone users in the United States. Such predictions are important tools for cellular telephone company executives and planners.

Before using mathematics to solve a real-world problem, we must usually set up a **mathematical model**, a mathematical description of the situation. In this chapter we look at some mathematics of *linear* models, which are used for data whose graphs can be approximated by straight lines. Linear models have an immense number of applications, because even when the underlying phenomenon is not linear, a linear model often provides an approximation that is sufficiently accurate and much simpler to use.

1.1 Slopes and Equations of Lines

APPLY IT **How fast has tuition at public colleges been increasing in recent years, and how well can we predict tuition in the future?**

In Example 13 of this section, we will answer these questions using the equation of a line.

There are many everyday situations in which two quantities are related. For example, if a bank account pays 6% simple interest per year, then the interest I that a deposit of P dollars would earn in one year is given by

$$I = 0.06 \cdot P, \qquad \text{or} \qquad I = 0.06P.$$

The formula $I = 0.06P$ describes the relationship between interest and the amount of money deposited.

Using this formula, we see, for example, that if $P = \$100$, then $I = \$6$, and if $P = \$200$, then $I = \$12$. These corresponding pairs of numbers can be written as **ordered pairs**, $(100, 6)$ and $(200, 12)$, whose order is important. The first number denotes the value of P and the second number the value of I.

Ordered pairs are graphed with the perpendicular number lines of a **Cartesian coordinate system**, shown in Figure 1.* The horizontal number line, or ***x*-axis**, represents the first components of the ordered pairs, while the vertical or ***y*-axis** represents the second components. The point where the number lines cross is the zero point on both lines; this point is called the **origin**.

Each point on the xy-plane corresponds to an ordered pair of numbers, where the x-value is written first. From now on, we will refer to the point corresponding to the ordered pair (x, y) as "the point (x, y)."

Locate the point $(-2, 4)$ on the coordinate system by starting at the origin and counting 2 units to the left on the horizontal axis and 4 units upward, parallel to the vertical axis. This point is shown in Figure 1, along with several other sample points. The number -2 is the ***x*-coordinate** and the number 4 is the ***y*-coordinate** of the point $(-2, 4)$.

The x-axis and y-axis divide the plane into four parts, or **quadrants**. For example, quadrant I includes all those points whose x- and y-coordinates are both positive. The quadrants are numbered as shown in Figure 1. The points on the axes themselves belong to no quadrant. The set of points corresponding to the ordered pairs of an equation is the **graph** of the equation.

The x- and y-values of the points where the graph of an equation crosses the axes are called the ***x*-intercept** and ***y*-intercept**, respectively.** See Figure 2.

*The name "Cartesian" honors René Descartes (1596–1650), one of the greatest mathematicians of the 17th century. According to legend, Descartes was lying in bed when he noticed an insect crawling on the ceiling and realized that if he could determine the distance from the bug to each of two perpendicular walls, he could describe its position at any given moment. The same idea can be used to locate a point in a plane.

**Some people prefer to define the intercepts as ordered pairs, rather than as numbers.

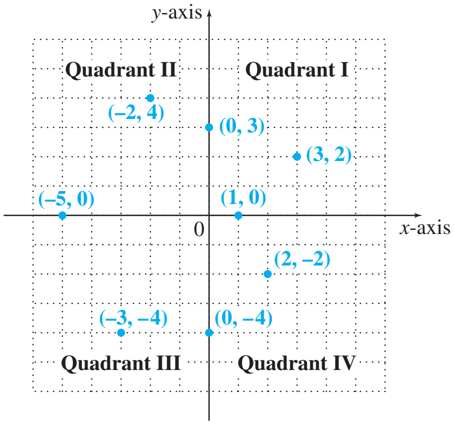


FIGURE 1

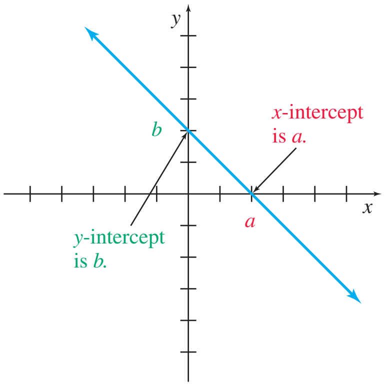


FIGURE 2

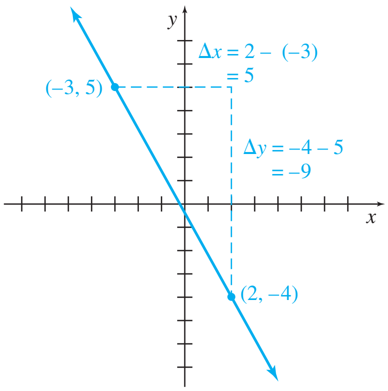


FIGURE 3

Slope of a Line

An important characteristic of a straight line is its *slope*, a number that represents the "steepness" of the line. To see how slope is defined, look at the line in Figure 3. The line passes through the points $(x_1, y_1) = (-3, 5)$ and $(x_2, y_2) = (2, -4)$. The difference in the two x-values,

$$x_2 - x_1 = 2 - (-3) = 5$$

in this example, is called the **change in x**. The symbol Δx (read "delta x") is used to represent the change in x. In the same way, Δy represents the **change in y**. In our example,

$$\begin{aligned}\Delta y &= y_2 - y_1\\ &= -4 - 5\\ &= -9.\end{aligned}$$

These symbols, Δx and Δy, are used in the following definition of slope.

Slope of a Nonvertical Line

The **slope** of a nonvertical line is defined as the vertical change (the "rise") over the horizontal change (the "run") as one travels along the line. In symbols, taking two different points (x_1, y_1) and (x_2, y_2) on the line, the slope is

$$\boldsymbol{m = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1},}$$

where $x_1 \neq x_2$.

By this definition, the slope of the line in Figure 3 is

$$m = \frac{\Delta y}{\Delta x} = \frac{-4 - 5}{2 - (-3)} = -\frac{9}{5}.$$

The slope of a line tells how fast y changes for each unit of change in x.

NOTE Using similar triangles, it can be shown that the slope of a line is independent of the choice of points on the line. That is, the same slope will be obtained for *any* choice of two different points on the line.

EXAMPLE 1 Slope

Find the slope of the line through each pair of points.

(a) $(7, 6)$ and $(-4, 5)$

SOLUTION Let $(x_1, y_1) = (7, 6)$ and $(x_2, y_2) = (-4, 5)$. Use the definition of slope.

$$m = \frac{\Delta y}{\Delta x} = \frac{5 - 6}{-4 - 7} = \frac{-1}{-11} = \frac{1}{11}$$

(b) $(5, -3)$ and $(-2, -3)$

SOLUTION Let $(x_1, y_1) = (5, -3)$ and $(x_2, y_2) = (-2, -3)$. Then

$$m = \frac{-3 - (-3)}{-2 - 5} = \frac{0}{-7} = 0.$$

Lines with zero slope are horizontal (parallel to the x-axis).

(c) $(2, -4)$ and $(2, 3)$

SOLUTION Let $(x_1, y_1) = (2, -4)$ and $(x_2, y_2) = (2, 3)$. Then

$$m = \frac{3 - (-4)}{2 - 2} = \frac{7}{0},$$

which is undefined. This happens when the line is vertical (parallel to the y-axis).

YOUR TURN 1 Find the slope of the line through $(1, 5)$ and $(4, 6)$.

TRY YOUR TURN 1

CAUTION The phrase "no slope" should be avoided; specify instead whether the slope is zero or undefined.

In finding the slope of the line in Example 1(a), we could have let $(x_1, y_1) = (-4, 5)$ and $(x_2, y_2) = (7, 6)$. In that case,

$$m = \frac{6 - 5}{7 - (-4)} = \frac{1}{11},$$

the same answer as before. The order in which coordinates are subtracted does not matter, as long as it is done consistently.

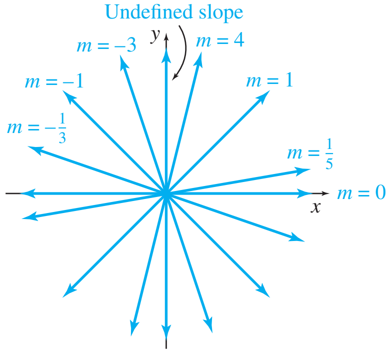


FIGURE 4

Figure 4 shows examples of lines with different slopes. Lines with positive slopes rise from left to right, while lines with negative slopes fall from left to right.

It might help you to compare slope with the percent grade of a hill. If a sign says a hill has a 10% grade uphill, this means the slope is 0.10, or 1/10, so the hill rises 1 foot for every 10 feet horizontally. A 15% grade downhill means the slope is -0.15.

Equations of a Line

An equation in two first-degree variables, such as $4x + 7y = 20$, has a line as its graph, so it is called a **linear equation**. In the rest of this section, we consider various forms of the equation of a line.

Suppose a line has a slope m and y-intercept b. This means that it passes through the point $(0, b)$. If (x, y) is any other point on the line, then the definition of slope tells us that

$$m = \frac{y - b}{x - 0}.$$

FOR REVIEW
For review on solving a linear equation, see Section R.4.

We can simplify this equation by multiplying both sides by x and adding b to both sides. The result is

$$y = mx + b,$$

which we call the *slope-intercept* form of a line. This is the most common form for writing the equation of a line.

Slope-Intercept Form

If a line has slope m and y-intercept b, then the equation of the line in **slope-intercept form** is

$$y = mx + b.$$

When $b = 0$, we say that y is **proportional** to x.

EXAMPLE 2 Equation of a Line

Find an equation in slope-intercept form for each line.

(a) Through $(0, -3)$ with slope $3/4$

SOLUTION We recognize $(0, -3)$ as the y-intercept because it's the point with 0 as its x-coordinate, so $b = -3$. The slope is $3/4$, so $m = 3/4$. Substituting these values into $y = mx + b$ gives

$$y = \frac{3}{4}x + (-3) = \frac{3}{4}x - 3.$$

(b) With x-intercept 7 and y-intercept 2

SOLUTION Notice that $b = 2$. To find m, use the definition of slope after writing the x-intercept as $(7, 0)$ (because the y-coordinate is 0 where the line crosses the x-axis) and the y-intercept as $(0, 2)$.

$$m = \frac{0 - 2}{7 - 0} = -\frac{2}{7}$$

Substituting these values into $y = mx + b$, we have

$$y = -\frac{2}{7}x + 2.$$

TRY YOUR TURN 2

YOUR TURN 2 Find the equation of the line with x-intercept -4 and y-intercept 6.

EXAMPLE 3 Finding the Slope

Find the slope of the line whose equation is $3x - 4y = 12$.

SOLUTION To find the slope, solve the equation for y.

$$3x - 4y = 12$$

$$-4y = -3x + 12 \qquad \text{Subtract } 3x \text{ from both sides.}$$

$$y = \frac{3}{4}x - 3 \qquad \text{Divide both sides by } -4.$$

The coefficient of x is $3/4$, which is the slope of the line. Notice that this is the same line as in Example 2(a).

TRY YOUR TURN 3

YOUR TURN 3 Find the slope of the line whose equation is $8x + 3y = 5$.

The slope-intercept form of the equation of a line involves the slope and the y-intercept. Sometimes, however, the slope of a line is known, together with one point (perhaps *not* the y-intercept) that the line passes through. The *point-slope form* of the equation of a line is used to find the equation in this case. Let (x_1, y_1) be any fixed point on the line, and let (x, y) represent any other point on the line. If m is the slope of the line, then by the definition of slope,

$$\frac{y - y_1}{x - x_1} = m,$$

or

$$y - y_1 = m(x - x_1). \qquad \text{Multiply both sides by } x - x_1.$$

Point-Slope Form

If a line has slope m and passes through the point (x_1, y_1), then an equation of the line is given by

$$y - y_1 = m(x - x_1),$$

the **point-slope form** of the equation of a line.

EXAMPLE 4 Point-Slope Form

Find an equation of the line that passes through the point $(3, -7)$ and has slope $m = 5/4$.

SOLUTION Use the point-slope form.

$$y - y_1 = m(x - x_1)$$

$$y - (-7) = \frac{5}{4}(x - 3) \qquad y_1 = -7, m = \tfrac{5}{4}, x_1 = 3$$

$$y + 7 = \frac{5}{4}(x - 3)$$

$$4y + 28 = 5(x - 3) \qquad \text{Multiply both sides by 4.}$$

$$4y + 28 = 5x - 15 \qquad \text{Distribute.}$$

$$4y = 5x - 43 \qquad \text{Subtract 28 from both sides.}$$

$$y = \frac{5}{4}x - \frac{43}{4} \qquad \text{Divide both sides by 4.}$$

FOR REVIEW
See Section R.4 for details on eliminating denominators in an equation.

The point-slope form also can be useful to find an equation of a line if we know two different points that the line goes through, as in the next example.

EXAMPLE 5 Point-Slope Form with Two Points

Find an equation of the line through $(5, 4)$ and $(-10, -2)$.

SOLUTION Begin by using the definition of slope to find the slope of the line that passes through the given points.

$$\text{Slope} = m = \frac{-2 - 4}{-10 - 5} = \frac{-6}{-15} = \frac{2}{5}$$

Either $(5, 4)$ or $(-10, -2)$ can be used in the point-slope form with $m = 2/5$. If $(x_1, y_1) = (5, 4)$, then

$$y - y_1 = m(x - x_1)$$

$$y - 4 = \frac{2}{5}(x - 5) \qquad y_1 = 4, m = \tfrac{2}{5}, x_1 = 5$$

$$y - 4 = \frac{2}{5}x - 2 \qquad \text{Distributive property}$$

$$y = \frac{2}{5}x + 2. \qquad \text{Add 4 to both sides.}$$

Check that the same result is found if $(x_1, y_1) = (-10, -2)$. TRY YOUR TURN 4

YOUR TURN 4 Find the equation of the line through $(2, 9)$ and $(5, 3)$. Put your answer in slope-intercept form.

EXAMPLE 6 Horizontal Line

Find an equation of the line through $(8, -4)$ and $(-2, -4)$.

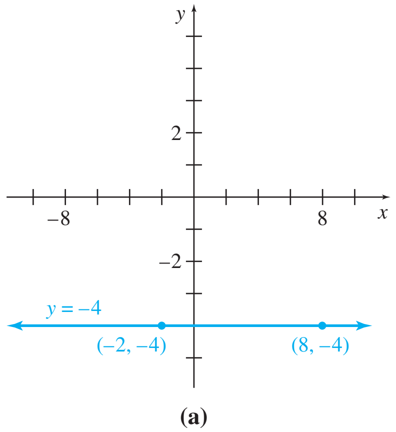
SOLUTION Find the slope.

$$m = \frac{-4 - (-4)}{-2 - 8} = \frac{0}{-10} = 0$$

Choose, say, $(8, -4)$ as (x_1, y_1).

$$y - y_1 = m(x - x_1)$$
$$y - (-4) = 0(x - 8) \qquad y_1 = -4, m = 0, x_1 = 8$$
$$y + 4 = 0 \qquad 0(x - 8) = 0$$
$$y = -4$$

Plotting the given ordered pairs and drawing a line through the points show that the equation $y = -4$ represents a horizontal line. See Figure 5(a). Every horizontal line has a slope of zero and an equation of the form $y = k$, where k is the y-value of all ordered pairs on the line.



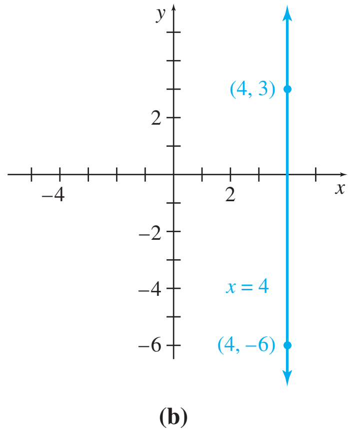


FIGURE 5

EXAMPLE 7 Vertical Line

Find an equation of the line through $(4, 3)$ and $(4, -6)$.

SOLUTION The slope of the line is

$$m = \frac{-6 - 3}{4 - 4} = \frac{-9}{0},$$

which is undefined. Since both ordered pairs have x-coordinate 4, the equation is $x = 4$. Because the slope is undefined, the equation of this line cannot be written in the slope-intercept form.

Again, plotting the given ordered pairs and drawing a line through them show that the graph of $x = 4$ is a vertical line. See Figure 5(b).

Slope of Horizontal and Vertical Lines

The slope of a horizontal line is 0.

The slope of a vertical line is undefined.

The different forms of linear equations discussed in this section are summarized below. The slope-intercept and point-slope forms are equivalent ways to express the equation of a nonvertical line. The slope-intercept form is simpler for a final answer, but you may find the point-slope form easier to use when you know the slope of a line and a point through which the line passes. The slope-intercept form is often considered the standard form. Any line that is not vertical has a unique slope-intercept form but can have many point-slope forms for its equation.

Equations of Lines

Equation	Description
$y = mx + b$	**Slope-intercept form:** slope m, y-intercept b
$y - y_1 = m(x - x_1)$	**Point-slope form:** slope m, line passes through (x_1, y_1)
$x = k$	**Vertical line:** x-intercept k, no y-intercept (except when $k = 0$), undefined slope
$y = k$	**Horizontal line:** y-intercept k, no x-intercept (except when $k = 0$), slope 0

Parallel and Perpendicular Lines

One application of slope involves deciding whether two lines are parallel, which means that they never intersect. Since two parallel lines are equally "steep," they should have the same slope. Also, two lines with the same "steepness" are parallel.

Parallel Lines

Two lines are **parallel** if and only if they have the same slope, or if they are both vertical.

EXAMPLE 8 Parallel Line

Find the equation of the line that passes through the point $(3, 5)$ and is parallel to the line $2x + 5y = 4$.

SOLUTION The slope of $2x + 5y = 4$ can be found by writing the equation in slope-intercept form. To put the equation in this form, solve for y.

$$2x + 5y = 4$$

$$y = -\frac{2}{5}x + \frac{4}{5} \qquad \text{Subtract } 2x \text{ from both sides and divide both sides by 5.}$$

This result shows that the slope is $-2/5$, the coefficient of x. Since the lines are parallel, $-2/5$ is also the slope of the line whose equation we want. This line passes through $(3, 5)$. Substituting $m = -2/5$, $x_1 = 3$, and $y_1 = 5$ into the point-slope form gives

$$y - y_1 = m(x - x_1)$$

$$y - 5 = -\frac{2}{5}(x - 3)$$

$$y - 5 = -\frac{2}{5}x + \frac{6}{5} \qquad \text{Distributive property}$$

$$y = -\frac{2}{5}x + \frac{6}{5} + 5 \cdot \frac{5}{5} \qquad \text{Add 5 to both sides and get a common denominator.}$$

$$y = -\frac{2}{5}x + \frac{31}{5}.$$

TRY YOUR TURN 5

YOUR TURN 5 Find the equation of the line that passes through the point $(4, 5)$ and is parallel to the line $3x - 6y = 7$. Put your answer in slope-intercept form.

As already mentioned, two nonvertical lines are parallel if and only if they have the same slope. Two lines having slopes with a product of -1 are perpendicular. A proof of this fact, which depends on similar triangles from geometry, is given as Exercise 43 in this section.

Perpendicular Lines

Two lines are **perpendicular** if and only if the product of their slopes is -1, or if one is vertical and the other horizontal.

EXAMPLE 9 Perpendicular Line

Find the equation of the line L passing through the point $(3, 7)$ and perpendicular to the line having the equation $5x - y = 4$.

SOLUTION To find the slope, write $5x - y = 4$ in slope-intercept form:

$$y = 5x - 4.$$

The slope is 5. Since the lines are perpendicular, if line L has slope m, then

$$5m = -1$$
$$m = -\frac{1}{5}.$$

Now substitute $m = -1/5$, $x_1 = 3$, and $y_1 = 7$ into the point-slope form.

$$y - 7 = -\frac{1}{5}(x - 3)$$

$$y - 7 = -\frac{1}{5}x + \frac{3}{5} \quad \text{Distribute.}$$

$$y = -\frac{1}{5}x + \frac{3}{5} + 7 \cdot \frac{5}{5} \quad \text{Add 7 to both sides and get a common denominator.}$$

$$y = -\frac{1}{5}x + \frac{38}{5}$$

TRY YOUR TURN 6

YOUR TURN 6 Find the equation of the line passing through the point $(3, 2)$ and perpendicular to the line having the equation $2x + 3y = 4$.

The next example uses the equation of a line to analyze real-world data. In this example, we are looking at how one variable changes over time. To simplify the arithmetic, we will *rescale* the variable representing time, although computers and calculators have made rescaling less important than in the past. Here it allows us to work with smaller numbers, and, as you will see, find the y-intercept of the line more easily. We will use rescaling on many examples throughout this book. When we do, it is important to be consistent.

EXAMPLE 10 Prevalence of Cigarette Smoking

In recent years, the percentage of the U.S. population age 18 and older who smoke has decreased at a roughly constant rate, from 20.9% in 2005 to 18.1% in 2012. ***Source: Centers for Disease Control and Prevention***.

(a) Find the equation describing this linear relationship.

SOLUTION Let t represent time in years, with $t = 0$ representing 2000. With this rescaling, the year 2005 corresponds to $t = 5$ and the year 2012 corresponds to $t = 12$. Let y represent the percentage of the population who smoke. The two ordered pairs representing the given information are then $(5, 20.9)$ and $(12, 18.1)$. The slope of the line through these points is

$$m = \frac{18.1 - 20.9}{12 - 5} = \frac{-2.8}{7} = -0.4.$$

This means that, on average, the percentage of the adult population who smoke is decreasing by about 0.4% per year.